# Are Hyperinflation Paths Learnable?*


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**Abstract**

Earlier studies of the seigniorage inflation model have found that the high-inflation steady state is not stable under learning. We reconsider this issue and analyze the full set of solutions for the linearized model. Our main focus is on stationary hyperinflationary paths near the high-inflation steady state. These paths are shown to be stable under least squares learning if agents can utilize contemporaneous data. In an economy with a mixture of agents, some of whom only have access to lagged data, stable hyperinflationary paths emerge only if the proportion of agents with access to contemporaneous data is sufficiently high.



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# 1 Introduction

The monetary inflation model, in which the demand for real balances depends negatively on expected inflation and the government uses seigniorage to fund in part its spending on goods, has two steady states and also perfect foresight paths that converge to the high inflation steady state.[1] These paths have occasionally been used as a model of hyperinflation, see e.g. (Fischer 1984), (Bruno 1989) and (Sargent and Wallace 1987). However, this approach remains controversial for several reasons. First, the high inflation steady state has "perverse" comparative static properties since an increase in seigniorage leads to lower steady state inflation. Second, recent studies of stability under learning of the high inflation steady state suggest that this steady state may not be a plausible equilibrium.

(Marcet and Sargent 1989) and (Evans, Honkapohja, and Marimon 2001) have shown that the high inflation steady state is unstable for various version of least squares learning. (Adam 2003) has obtained the same result for a sticky price version of the monetary inflation model with monopolistic competition. (Arifovic 1995) has examined the model under genetic algorithm learning and the economy appears always to converge to the steady state with low, rather than high inflation. Experimental work by (Marimon and Sunder 1993) also comes to the conclusion that the high inflation steady state is not a plausible outcome in the monetary inflation model.

The instability result for the high inflation steady state under learning has been derived under a particular assumption about the information sets that agents are assumed to have. (Van Zandt and Lettau 2003) raise questions about the timing and information sets in the context of learning steady states. They show that, under what is often called constant gain learning, the high inflation steady state in the Cagan model can be stable under learning with specific informational assumptions.[2] Under the more standard decreasing gain learning the high inflation steady state is found to be stable only if inflation is estimated by a regression of the price level on its lagged value (without intercept) and the current price level is both included as part of the information set and used to update current parameter estimates as well.

[1] The model is also called the Cagan model after (Cagan 1956).

[2] However, constant gain learning is most natural in nonstochastic models, since otherwise convergence to rational expectations is precluded. In this paper we allow for intrinsic random shocks and thus use "decreasing gain" algorithms, consistent with least squares learning.

The theoretical learning stability results of both (Marcet and Sargent 1989) and (Van Zandt and Lettau 2003) are for nonstochastic models and examine only the learnability of inflation steady states.[3] However, because the high inflation steady state is indeterminate, there exists a multiplicity of solutions taking different forms. In stochastic models, near the high inflation steady state the solutions include stochastically stationary first order autoregressive solutions driven by the fundamental shocks and also more general solutions that depend on sunspots. The central goal of the current paper is to assess the stability under least squares learning of the entire class of rational expectations (RE) solutions.[4] In doing so we pay careful attention to the information sets of the agents.

The monetary inflation model, like that of (Duffy 1994), has the important feature that the temporary equilibrium inflation rate in period $t$ depends on the private agents' one-step ahead forecasts of inflation made in two successive periods, $t-1$ and $t$. Except for some partial results in (Duffy 1994) and (Adam 2003), the different types of rational expectations equilibria (REE) in such "mixed dating models" have not been examined for stability under least squares learning. We show that stationary AR(1) paths, as well associated sunspot equilibria around an indeterminate steady state, such as the high inflation steady state, are stable under learning when agents have access to contemporaneous data of endogenous variables. However, this result is sensitive to the information assumption. If the economy has sufficiently many agents who base their forecasts only on lagged information, then the results are changed and the equilibria just mentioned become unstable under learning.

Although our main interest is theoretical, in the last section of the paper we discuss the empirical implications of our stable hyperinflationary paths. (Marcet and Nicolini 2003) review the stylized facts of hyperinflationary episodes for Latin American countries and argue that, in particular, there is a low correlation between seigniorage and inflation. To account for this, they construct a model of recurrent hyperinflations based upon a low inflation steady state that is stable under learning and occasional trajectories into the unstable region near the high inflation steady state, followed by a policy switch that stabilizes inflation back to low levels. Our finding of learnable

[3](Evans, Honkapohja, and Marimon 2001) analyze learning of stochastic steady states.

[4]In a related but different model, (Duffy 1994) showed the possibility of expectationally stable nonstochastic dynamic paths near an indeterminate steady state.

paths near the high inflation steady state raises the question of whether the correlation of seigniorage and inflation along these paths is consistent with the stylized facts. We show that the correlation can take any value, depending on the specific solution within the class of stable hyperinflationary REE, and that a dependence on sunspots reduces the magnitude of the correlation. This suggests that extensions of this model do have potential as an explanation for hyperinflationary episodes.

# 2 The hyperinflation model

Consider an overlapping generations economy where agents born after period zero live for two periods. An agent of generation $t \geq 1$ has a two-period endowment of a unique perishable good $(w_{t,0}, w_{t,1}) = (2\psi_0, 2\psi_1)$, $\psi_0 > \psi_1 > 0$, with preferences over consumption given by $u(c_{t,0}, c_{t,1}) = \ln(c_{t,0}) + \ln(c_{t,1})$ where the second subscript indexes the periods in the agent's life. The agent of the initial generation only lives for one period, has preferences $u(c_{0,1}) = \ln c_{0,1}$, and is endowed with $2\psi_1$ units of the consumption good and $M_0$ units of fiat money, which is the only means of saving.[5]

Let $P_t$ denote the money price of the consumption good in period $t$ and use $m_t = \frac{M_t}{P_t}$ to denote real money balances. Utility maximization by agents then implies that real money demand of generation $t$ is given by

$$m_t^d = \psi_0 - \psi_1 E_t^* x_{t+1} \tag{1}$$

where $x_{t+1} = \frac{P_{t+1}}{P_t}$ denotes the inflation factor from $t$ to $t+1$. Here $E_t^* x_{t+1}$ denotes expected inflation, which we do not restrict to be fully rational (we will reserve $E_t x_{t+1}$ for rational expectations).[6]

Real money supply $m_t^s$ is given by

$$m_t^s = \frac{m_{t-1}}{x_t} + g + v_t$$

where $g$ is the mean value of real seigniorage, and $v_t$ is a stochastic seigniorage term assumed to be white noise with small bounded support and zero mean.[7]

[5] This framework, which is standard, was used by (Evans, Honkapohja, and Marimon 2001) and (Marcet and Nicolini 2003).

[6] The money demand function (1) can also be viewed as a log-version of the (Cagan 1956) demand function.

[7] More generally the monetary shock could be allowed to be a martingale difference sequence with small bounded support.

This formulation of the seigniorage equation is standard, see e.g. (Sargent and Wallace 1987), and simply states that government purchases of goods $g + v_t$ are financed by issuing fiat money. It would also be straightforward to allow for a fixed amount of government purchases financed by lump-sum taxes.

Market clearing in all periods implies that

$$x_t = \frac{\psi_0 - \psi_1 E^*_{t-1} x_t}{\psi_0 - \psi_1 E^*_t x_{t+1} - g - v_t}. \tag{2}$$

Provided

$$g < g^{\max} = \left(\sqrt{\psi_0} - \sqrt{\psi_1}\right)^2,$$

there exist two noisy steady states, with different mean inflation rates $x$, given by the quadratic

$$\psi_1 x^2 - (\psi_1 + \psi_0 - g)x + \psi_0 = 0. \tag{3}$$

We denote the low inflation steady state by $x^l$ and the high inflation steady state by $x^h$. Throughout the paper we will assume that $g < g^{\max}$ so that both steady states exist. As shown in Appendix A.1, the low inflation steady state is locally unique, while there is a continuum of stationary REE in a neighborhood of the high inflation steady state.

The model (2) can be linearized around either steady state, leading to a reduced form that fits into a general mixed dating expectations model taking the form

$$x_t = \alpha + \beta_1 E^*_t x_{t+1} + \beta_0 E^*_{t-1} x_t + u_t, \tag{4}$$

where $u_t$ is a positive scalar times $v_t$. It is convenient to study learning within the context of the linearized model (4), and this has the advantage that our results can also be used to discuss related models with the same linearized reduced form, e.g. the one of (Duffy 1994).

The linearization of the hyperinflation model is discussed in detail in Appendix A.1. We here note that equation (2) implies $\beta_1 > 0$ and $\beta_0 < 0$ for the linearization at either steady state. Furthermore the coefficients $(\alpha, \beta_0, \beta_1)$ at either steady state are functions of the parameters $\omega$ and $\xi$ only, where

$$\omega = \frac{\psi_1}{\psi_0} \quad \text{and} \quad \xi = \frac{g}{g^{\max}}.$$

# 3 The mixed dating model

We start by determining the complete set of rational expectations equilibria for model (4). These can be obtained as follows. In a rational expectations equilibrium (REE) the forecast error

$$\eta_t = x_t - E_{t-1}x_t$$

is a martingale difference sequence (MDS), which together with (4) implies that

$$x_t = \alpha + \beta_1(x_{t+1} - \eta_{t+1}) + \beta_0(x_t - \eta_t) + u_t.$$

Solving for $x_{t+1}$ and lagging the equation by one period delivers

$$x_t = -\beta_1^{-1}\alpha + \beta_1^{-1}(1-\beta_0)x_{t-1} + \eta_t + \beta_1^{-1}\beta_0\eta_{t-1} - \beta_1^{-1}u_{t-1}$$

One can decompose the arbitrary MDS $\eta_t$ into a component that is correlated with $u_t$ and an orthogonal sunspot $\eta_t'$ :

$$\eta_t = \gamma_0 u_t + \gamma_1 \eta_t'$$

The sunspot $\eta_t'$ is again a MDS. Moreover, since $\eta_t$ is an arbitrary MDS, the coefficients $\gamma_0$ and $\gamma_1$ are free to take on any values. This delivers the full set of rational expectations solutions for the model:

$$x_t = -\frac{\alpha}{\beta_1} + \frac{(1-\beta_0)}{\beta_1}x_{t-1} + \gamma_0 u_t + \frac{(\beta_0\gamma_0 - 1)}{\beta_1}u_{t-1} + \gamma_1\eta_t' + \frac{\beta_0\gamma_1}{\beta_1}\eta_{t-1}' \quad (5)$$

Since $\gamma_0$ and $\gamma_1$ are arbitrary there is a continuum of ARMA(1,1) sunspot equilibria.

For $\gamma_0 = 1$ and $\gamma_1 = 0$ we obtain the stochastic steady state solution

$$x_t = \alpha(1-\beta_1-\beta_0)^{-1} + u_t, \quad (6)$$

while setting $\gamma_0 = \beta_0^{-1}$ and $\gamma_1 = 0$ yields an AR(1) solution

$$x_t = -\beta_1^{-1}\alpha + \beta_1^{-1}(1-\beta_0)x_{t-1} + \beta_0^{-1}u_t \quad (7)$$

A third special case of interest arises when $\gamma_1 = 0$ but $\gamma_0 \neq \beta_0^{-1}$ yielding

$$x_t = -\frac{\alpha}{\beta_1} + \frac{(1-\beta_0)}{\beta_1}x_{t-1} + \gamma_0 u_t + \frac{(\beta_0\gamma_0 - 1)}{\beta_1}u_{t-1}, \quad (8)$$

which is a continuum of ARMA(1,1) equilibria that does not depend on extraneous sunspots. One might call the solutions (8) "intrinsic nonfundamental equilibria" since they are driven only by intrinsic random shocks but are nonfundamental in the sense that they do not depend on a minimal number of state variables.[8]

For the hyperinflation model, stability of steady state solutions (6) has been studied in (Marcet and Sargent 1989), (Evans, Honkapohja, and Marimon 2001) and (Van Zandt and Lettau 2003). In the current paper, we examine stability under learning of the full set of ARMA(1,1) solutions (5). As shown in Appendix A.1, the ARMA(1,1) solutions near the high inflation steady state are stationary, since $0 < \beta_1^{-1}(1 - \beta_0) < 1$ for the linearization at $x^h$. We demonstrate that these solutions are stable under learning when all agents have full current information. These stability results apply also to the solution special cases (7) and (8). We additionally examine how stability is affected by the information sets of the agents, and in particular by the possibility that a proportion of agents do not have access to full current information. Our focus is thus on the stability under learning of the stationary solutions, near the high inflation steady state, other than the stochastic steady state itself, and on the robustness of stability to the presence of a mixture of agents with differing information sets.

# 4 Learning with full current information

We first consider the situation where agents have information about all variables up to time $t$ and wish to learn the parameters of the rational expectations solution (5). As is well-known, the conditions for local stability under least squares learning are given by expectational stability (E-stability) conditions. Therefore, we first discuss the E-stability conditions for the REE, after which we take up real time learning.

## 4.1 E-stability

Agents' perceived law of motion (PLM) of the state variable $x_t$ is given by

$$x_t = a + bx_{t-1} + cu_{t-1} + d\eta'_{t-1} + \zeta_t \tag{9}$$

[8] This is in contrast to the "minimal state variable" (MSV) solution (6). See (McCallum 1983) for a discussion of MSV solutions.

where the parameters $(a, b, c, d)$ are not known to the agent but are estimated by least-squares, and $\zeta_t$ represents unforecastable noise.

Substituting the expectations generated by the PLM (9) into the model (4) delivers the actual law of motion (ALM) for the state variable $x_t$:

$$\begin{aligned} x_t &= (1-\beta_1 b)^{-1}\left[\alpha + (\beta_1+\beta_0)a\right] \\ &\quad + (1-\beta_1 b)^{-1}\left[\beta_0 b x_{t-1} + (1+\beta_1 c)u_t + \beta_0 c u_{t-1} + \beta_1 d\eta'_t + \beta_0 d\eta'_{t-1}\right] \end{aligned} \tag{10}$$

The map from the parameters in the PLM to the corresponding parameters in the ALM, the T-map in short, is given by

$$\begin{aligned} a &\to \frac{\alpha+(\beta_1+\beta_0)a}{1-\beta_1 b},\ b \to \frac{\beta_0 b}{1-\beta_1 b} \\ c &\to \frac{\beta_0 c}{1-\beta_1 b},\ d \to \frac{\beta_0 d}{1-\beta_1 b} \end{aligned}$$

Since the variables entering the ALM also show up in the PLM, the fixed points of the T-map are rational expectations equilibria. Furthermore, as is easy to verify, all REE's are also fixed points of the T-map.

Local stability of a REE under least squares learning of the parameters in (9) is determined by the stability of the differential equation

$$\frac{d(a,b,c,d)}{d\tau} = T(a,b,c,d) - (a,b,c,d) \tag{11}$$

at the REE. This is known as the E-stability differential equation, and the connection to least squares learning is discussed more generally and at length in (Evans and Honkapohja 2001). If an REE is locally asymptotically stable under (11) then the REE is said to be "expectationally stable" or "E-stable."

Equation (11) is stable if and only if the eigenvalues of

$$DT = \begin{pmatrix} \frac{\beta_1+\beta_0}{1-\beta_1 b} & \frac{-(\alpha+(\beta_1+\beta_0)a)\beta_1}{(1-\beta_1 b)^2} & 0 & 0 \\ 0 & \frac{\beta_0}{(1-\beta_1 b)^2} & 0 & 0 \\ 0 & -\frac{\beta_1\beta_0 c}{(1-\beta_1 b)^2} & \frac{\beta_0}{1-\beta_1 b} & 0 \\ 0 & -\frac{\beta_1\beta_0 d}{(1-\beta_1 b)^2} & 0 & \frac{\beta_0}{1-\beta_1 b} \end{pmatrix} \tag{12}$$

have real parts smaller than 1 at the REE. At the REE we have

$$a = -\beta_1^{-1}\alpha \tag{13}$$

$$b = \beta_1^{-1}(1-\beta_0) \tag{14}$$

$$c, d : arbitrary$$

and the eigenvalues of $DT$ are given by:

$$\lambda_1 = 1 + \frac{\beta_1}{\beta_0};\ \lambda_2 = \frac{1}{\beta_0};\ \lambda_3 = 1;\ \lambda_4 = 1$$

The eigenvectors corresponding to the last two eigenvalues are those pointing into the direction of $c$ and $d$, respectively. As one would expect, stability in the point-wise sense cannot hold for these parameters and in contexts such as these, E-stability is defined relative to the whole class of ARMA equilibria. A class of REE is then said to be E-stable if the dynamics under (11) converge to some member of the class from all initial points sufficiently near the class. We can then summarize the preceding analysis:

**Proposition 1** *If $\beta_1 > 0$ and $\beta_0 < 0$ or if $\beta_1 < 0$ and $\beta_0 > 1$ the set of ARMA(1,1)-REE is E-stable.*

It is easily seen that when the conditions of Proposition 1 hold, the solution special cases (7) and (8) are also E-stable under their corresponding PLMs.

Figure 1 illustrates these conditions in the $(\beta_0, \beta_1)$-space. The light grey region indicates parameter values for which the ARMA equilibria are E-stable but explosive. If $\beta_1$ and $\beta_0$ lies in the black region, then the ARMA equilibria are both E-stable and stochastically stationary.

FIGURE 1 HERE

Since $\beta_1 > 0$ and $\beta_0 < 0$ for the high steady state in the hyperinflation model, Proposition 1 implies that the set of stationary ARMA(1,1)-REE is E-stable.

We note that Proposition 1 applies to any model with reduced form (4). In particular, in the model of (Duffy 1994) we have $-\beta_1 = \beta_0 > 1$, and thus this proposition confirms his E-stability result for the stationary AR(1) solutions and, more generally, proves E-stability for stationary ARMA(1,1) sunspot solutions.

Since steady states are also fixed points of the T-map, the preceding E-stability analysis can be applied to these REE. Note that this allows for PLMs that are "overparameterized" relative to the steady state REE. We have:

**Remark:** *A steady state $(\alpha(1-\beta_1-\beta_0)^{-1}, 0, 0, 0)$ is E-stable when agents use the PLM (9) if $\beta_0 < 1$ and $\beta_0 + \beta_1 < 1$.*

## 4.2 Real time learning

Next we consider real time learning of the set of ARMA equilibria (5). This section shows that stochastic approximation theory can be applied to show convergence of least squares learning when the PLM of the agents has an AR(1) form and the economy can converge to the AR(1) equilibrium (7). For technical reasons the stochastic approximation tools cannot be applied for the continuum of ARMA(1,1)-REE. Therefore, real time learning of the class (5) REE will be considered in section 6.3 using simulations.

Assume first that agents have the PLM of AR(1) form, i.e.

$$x_t = a + bx_{t-1} + \zeta_t. \tag{15}$$

The parameters $a$ and $b$ are updated using recursive least squares using data through period $t$, so that the forecasts are given by

$$E_t^* x_{t+1} = a_t + b_t x_t,$$
$$E_{t-1}^* x_t = a_{t-1} + b_{t-1} x_{t-1}.$$

Substituting these forecasts into (4) yields the ALM

$$x_t = \frac{\alpha + \beta_0 a_{t-1} + \beta_1 a_t}{1 - \beta_1 b_t} + \frac{\beta_0 b_{t-1}}{1 - \beta_1 b_t} x_{t-1} + \frac{1}{1 - \beta_1 b_t} u_t. \tag{16}$$

Parameter updating is done using recursive least squares i.e.

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \vartheta_t R_t^{-1} (x_{t-1} - a_{t-1} - b_{t-1} x_{t-2}) \begin{pmatrix} 1 \\ x_{t-2} \end{pmatrix}, \tag{17}$$

where $R_t$ is the matrix of second moments, which will be explicitly specified in the Appendix, and $\vartheta_t$ is the gain sequence, which is a decreasing sequence such as $t^{-1}$.[9] In Appendix A.2 we prove the following result:

**Proposition 2** *The AR(1) equilibrium of model (4) is stable under least squares learning (17) if the model parameters satisfy the E-stability conditions given in Proposition 1.*

Since E-stability governs the stability of the AR(1) solution under least squares learning, the stationary AR(1) solutions in the hyperinflation model are learnable. The same result holds for the AR(1) solution in a stochastic version of the model of (Duffy 1994).

[9] See Chapter 2 of (Evans and Honkapohja 2001) for the recursive formulation of least squares estimation.

# 5 Learning without observing current states

The observability of current states, as assumed in the previous section, introduces a simultaneity between expectations and current outcomes. Technically this is reflected in $x_t$ appearing on both sides of the equation when substituting the PLM (9) into the model (4). To obtain the ALM one first has to solve this equation for $x_t$. Although this is straightforward mathematically, it is not clear what economic mechanism would ensure consistency between $x_t$ and the expectations based on $x_t$. Moreover, in the non-linear formulation there may even exist multiple mutually consistent price and price expectations pairs, as pointed out in (Adam 2003).

To study the role of the precise information assumption, we introduce a fraction of agents who cannot observe the current state $x_t$. Such agents in effect must learn to make forecasts that are consistent with current outcomes, which allows us to consider the robustness of the preceding results. Thus suppose that a share $\lambda$ of agents has information set

$$H_t' = \sigma(u_t, u_{t-1}, \ldots, \eta_t', \eta_{t-1}', \ldots, x_{t-1}, x_{t-2} \ldots)$$

and cannot observe the current state $x_t$. Let the remaining agents have the "full $t$"-information set

$$H_t = \sigma(u_t, u_{t-1}, \ldots, \eta_t', \eta_{t-1}', \ldots, x_t, x_{t-1} \ldots)$$

Expectations based on $H_t'$ are denoted by $E_t'^*[\cdot]$ and expectations based on $H_t$ by $E_t^*[\cdot]$.

With the relevant economic expectations given by the average expectations across agents, the economic model (4) can now be written as

$$\begin{aligned} x_t = \alpha &+ \beta_1 \left((1-\lambda)E_t^*\left[x_{t+1}\right] + \lambda E_t'^*\left[x_{t+1}\right]\right) \\ &+ \beta_0 \left((1-\lambda)E_{t-1}^*\left[x_t\right] + \lambda E_{t-1}'^*\left[x_t\right]\right) + u_t \end{aligned}$$

As before, the PLM of agents with information set $H_t$ will be given by

$$x_t = a_2 + b_2 x_{t-1} + c_2 u_{t-1} + d_2 \eta_{t-1}' + \zeta_t$$

while the PLM for agents with information set $H_t^{'}$ is given by

$$x_t = a_1 + b_1 x_{t-1} + e_1 u_t + c_1 u_{t-1} + f_1 \eta_t' + d_1 \eta_{t-1}' + \zeta_t^{'}.$$

Here $\zeta_t$ and $\zeta_t'$ represent zero mean disturbances that are uncorrelated with all variables in the respective information sets. Since agents with information set $H_t^{'}$ do not know $x_t$, they must first forecast $x_t$ to be able to forecast $x_{t+1}$. The forecast of $x_t$ depends on the current shocks $u_t$ and $\eta_t'$, which implies that these agents must estimate $e_1$ and $f_1$ to be able to forecast.

Agents' expectations are now given by

$$\begin{aligned}
E_t^*[x_{t+1}] &= a_2 + b_2 x_t + c_2 u_t + d_2 \eta_t' \\
E_t'^*[x_{t+1}] &= a_1 + b_1 E_t'^*[x_t] + c_1 u_t + d_1 \eta_t' \\
&= a_1 + b_1 \left[a_1 + b_1 x_{t-1} + e_1 u_t + c_1 u_{t-1} + f_1 \eta_t' + d_1 \eta_{t-1}'\right] + c_1 u_t + d_1 \eta_t' \\
&= a_1(1+b_1) + b_1^2 x_{t-1} + (b_1 e_1 + c_1) u_t \\
&\quad + b_1 c_1 u_{t-1} + (b_1 f_1 + d_1)\eta_t' + b_1 d_1 \eta_{t-1}'
\end{aligned}$$

and the implied ALM can be written as

$$z_t = A + B z_{t-1} + C \begin{pmatrix} u_t \\ \eta_t' \end{pmatrix} \tag{18}$$

where $z_t = (x_t, x_{t-1}, u_t, u_{t-1}, \eta_t', \eta_{t-1}')$ and where the expressions for $A$, $B$, and $C$ can be found in Appendix A.3.1.

It is important to note that the ALM is an ARMA(2,2) process and therefore of higher order than agents' PLM. This is due to the presence of agents with $H_t'$ information. These agents use variables dated $t-2$ to forecast $x_t$. This feature has several important implications. First, while the T-map is given by the coefficients showing up in the ARMA(2,2)-ALM (18), calculating the fixed points of the learning process now requires us to project the ARMA(2,2)-ALM back onto the ARMA(1,1) parameter space. Second, it might appear that the resulting fixed points of the T-map would not constitute rational expectations equilibria, but rather what have been called "restricted perceptions equilibria' (RPE). RPE have the property that agents' forecasts are optimal within the class of PLMs considered by agents, but not within a more general class of models.[10]

Because our agents estimate ARMA(1,1) models, and under the current information assumptions ARMA(1,1) PLMs generate ARMA(2,2) ALMs, there is clearly the possibility that convergence will be to an RPE that is not an REE. However, as we will show below, convergence will be to an

[10] The issue of projecting a higher-order ALM back to a lower-order PLM first arose in (Sargent 1991). Sargent's "reduced-order" equilibrium is a particular form of RPE.

ARMA(2,2) process that can be regarded as an overparameterized ARMA(1,1) REE. Therefore, the misspecification by agents is transitional and disappears asymptotically.

The projection of the ARMA(2,2)-ALM on the ARMA(1,1)-PLM is obtained as follows. Under the assumption that $z_t$ is stationary equation (18) implies

$$vec(var(z_t)) = (I - B \otimes B)^{-1} vec\left(C var\begin{pmatrix} u_t \\ \eta'_t \end{pmatrix} C'\right) \tag{19}$$

Using the covariances in (19) one can express the least squares estimates as

$$T\begin{pmatrix} b_i \\ e_i \\ c_i \\ f_i \\ d_i \end{pmatrix} = var\begin{pmatrix} x_{t-1} \\ u_t \\ u_{t-1} \\ \eta'_t \\ \eta'_{t-1} \end{pmatrix}^{-1} cov\left(\begin{pmatrix} x_{t-1} \\ u_t \\ u_{t-1} \\ \eta'_t \\ \eta'_{t-1} \end{pmatrix}, x_t\right)$$

The estimate for the constant is

$$\begin{aligned} T(a_i) &= (1 - b_i)E(x_t) \\ &= (1 - b_i)\frac{A_{11}}{1 - \frac{1}{1/\beta_1 - (1-\lambda)b_2}(B_{11} + B_{12})} \end{aligned}$$

where $A_{11}$, $B_{11}$, and $B_{12}$ are elements of the ALM coefficients $A$ and $B$, as given in Appendix A.3.1. This completes the projection of the ARMA(2,2)-ALM onto the ARMA(1,1)-PLM.

Using Mathematica one can then show that the following parameters are fixed points of the T-map:

$$\begin{aligned} &(a_1, b_1, e_1, c_1, f_1, d_1, a_2, b_2, c_2, d_2) \\ &= (-\alpha/\beta_1, 1/\beta_1 - \rho, \gamma_0, \gamma_0\rho - 1/\beta_1, \gamma_1, \gamma_1\rho, -\alpha/\beta_1, 1/\beta_1 - \rho, \gamma_0\rho - 1/\beta_1, \gamma_1\rho) \end{aligned} \tag{20}$$

where

$$\rho = \frac{\beta_0}{\beta_1}, \ \gamma_0, \gamma_1 : \text{arbitrary constants.}$$

Note that the PLMs of agents with information set $H_t$ and $H'_t$ is the same (up to the coefficients showing up in front of the additional regressors of $H'_t$-agents). This is not surprising since agents observe the same variables and estimate effectively the same PLMs.

It might appear surprising that the PLM-parameters in (20) are independent of the share $\lambda$ of agents with information set $H'_t$. One might expect that the value of $\lambda$ would affect the importance the second lags in the ALM (20) and therefore influence the projection of the ARMA(2,2)-ALM onto the PLMs. However, it can be shown that this is not true at the fixed point (20). Calculating the ALM implied by the fixed point (20) yields:

$$A(L)\left[1+(-\frac{1}{\beta_1}+\rho)L\right]x_t = A(L)\left[\gamma_0+(-\frac{1}{\beta_1}+\gamma_0\rho)L\right]u_t \qquad (21)$$
$$+\,A(L)\left[\gamma_1+\gamma_1\rho L\right]\eta'_t + A$$

where

$$A(L) = \frac{-\beta_1\rho-\lambda+\beta_1\rho\lambda}{\beta_1\rho(\lambda-1)-\lambda} + \frac{-\rho\lambda+\beta_1\rho^2\lambda}{\beta_1\rho(\lambda-1)-\lambda}L$$
$$A = \frac{\alpha\left((1+\rho)\lambda-\beta_1\rho(-1+\lambda+\rho\lambda)\right)}{\beta_1(\beta_1\rho(\lambda-1)-\lambda)}.$$

The ARMA(2,2)-ALM (18) has a common factor in the lag polynomials. Canceling the common factor $A(L)$ in (21) gives the ARMA(1,1)-REE (5). From $\rho = \frac{\beta_0}{\beta_1}$ it can be seen that the resulting ARMA(1,1) process is precisely the ARMA(1,1) REE (5).

To summarize the preceding argument, the ALM is a genuine ARMA(2,2) process during the learning transition and this is underparameterized by the agents estimating an ARMA(1,1). However, provided learning converges, this misspecification becomes asymptotically negligible.

As in the case of the ARMA(1,1)-REE, E-stability of the ARMA(1,1) fixed points are determined by the eigenvalues of the matrix

$$\frac{dT}{d(a_1,b_1,e_1,c_1,f_1,d_1,a_2,b_2,c_2,d_2)} \qquad (22)$$

evaluated at the fixed points.

As a first application of our setting, we consider the model of (Duffy 1994), which depends on a single parameter because $-\beta_1=\beta_0>1$. Using Mathematica to derive analytical expressions for the eigenvalues of (22), one can show that a necessary condition for E-stability is given by

$$\lambda < \frac{(\beta_1)^2}{2(\beta_1)^2-1}. \qquad (23)$$

Thus, in this model the ARMA(1,1)-REE become unstable if a high enough share of agents does not observe current endogenous variables.

We now turn to our main application, i.e. the hyperinflation model.

# 6 The hyperinflation model reconsidered

As discussed in the introduction, the low inflation steady state is learnable, and the high inflation steady state is not, under most assumptions concerning learning. In the setup considered here, the Remark at the end of Section 4.1 yields this result with full current information since $\beta_0 + \beta_1 < 1$ and $\beta_0 < 1$ at $x^l$ whereas $\beta_0 + \beta_1 > 1$ at $x^h$.[11] The contribution of the current paper is the discovery that a class of stationary solutions near $x^h$ are stable under least squares learning if all agents use full current information, and we now examine the robustness of this result for different values of $\lambda > 0$.

Consider the stability of the ARMA(1,1)-REE in the hyperinflation model when a share $\lambda$ of agents has information $H_t'$ and the remaining agents have full information $H_t$. We first examine the case of small amounts of seigniorage $\xi \to 0$, for which the expressions for the linearization coefficients and the equilibrium coefficients become particularly simple. We then present some results for the general case $\xi > 0$.

## 6.1 Small amounts of seigniorage

The linearization coefficients of the hyperinflation model for $\xi \to 0$ are given by

$$\lim_{\xi \to 0} \alpha = \frac{1}{\omega}, \ \lim_{\xi \to 0} \beta_1 = +\infty, \ \lim_{\xi \to 0} \rho = \frac{\beta_0}{\beta_1} = -\omega.$$

From equation (20) it then follows that in the ARMA(1,1)-REE the coefficients are given by

$$\begin{aligned}
&(a_1, b_1, e_1, c_1, f_1, d_1, a_2, b_2, c_2, d_2) \\
&= (0, \omega, \gamma_0, -\gamma_0\omega, \gamma_1, -\gamma_1\omega, 0, \omega, -\gamma_0\omega, -\gamma_1\omega).
\end{aligned}$$

E-stability of the ARMA(1,1)-REE is determined by the eigenvalues of the T-map. Analytical expressions for the eigenvalues are given in Appendix

[11] For steady state PLMs $x_t = a + u_t$ these stability results hold regardless of the information sets of the agents. The exceptions to the stability results noted by (Van Zandt and Lettau 2003) do not arise in our framework.

A.3.2. Four of these eigenvalues are equal to zero. Two eigenvalues are equal to one. The latter correspond to the eigenvectors pointing into the direction of the arbitrary constants $\gamma_0$ and $\gamma_1$. The remaining four eigenvalues $s_i$ ($i = 1, \ldots, 4$) are functions of $\omega$ and $\lambda$, and we compute numerical stability results.

FIGURE 2 HERE

For $\lambda$ values lying above the line shown in Figure 2 the ARMA(1,1) class of REE is E-unstable. A sufficient condition for instability is $\lambda > 1/2$ (since then $s_1 > 1$).

## 6.2 The intermediate and large deficit case

Using the analytical expressions for the eigenvalues of the matrix (22) we used numerical methods to determine the critical share $\lambda$ for which the ARMA(1,1)-REE becomes E-unstable for positive values of the deficit share $\xi$. Figure 3 displays the critical $\lambda$ values for $\xi$ values of 0.2, 0.5, and 0.95, respectively. For $\lambda$ values lying above the lines shown in these figures, the ARMA(1,1) class of REE is E-unstable. For $\lambda$ values below these lines the equilibria remain E-stable.

FIGURE 3 HERE

The figure suggests that $\lambda > 0.5$ continues to be a sufficient condition for E-instability of the ARMA(1,1) REE. However, critical values for $\lambda$ appear generally to be smaller than 0.5, with critical values significantly lower if $\omega$ is small and $\xi$ is high. Moreover, when $\omega = 0$ and $\xi \to 1$, these equilibria become unstable even if an arbitrarily small share of agents does not observe the current values of $x_t$.

## 6.3 Real time learning

Because formal real time learning results cannot be proved for the ARMA(1,1) sunspot solutions, we here present simulations of the model under learning. These indicate that the E-stability results do indeed provide the stability conditions of this class of solutions under least-squares learning. In the illustrative simulations we set $\beta_1 = 2$ and $\beta_0 = -0.5$ and $\alpha = 0$. For these reduced form parameters the values of $a$ and $b$ at the ARMA(1,1) REE are $a = 0$ and $b = 0.75$. For these reduced form parameters the ARMA(1,1) REE are E-stable for $\lambda = 0$, see Figure 1, and convergent parameter paths are indeed obtained under recursive least squares learning. The parameter estimates for a typical simulation, shown in Figures 4 and 5, are converging toward equilibrium values of the set of ARMA(1,1) REE.

FIGURES 4 THROUGH 7 HERE

In Figures 6 and 7 the share of agents with information set $H_t'$ is increased to $\lambda = 0.5$ and the ARMA(1,1) sunspot equilibria become unstable under learning: $a_t$, $b_t$ are clearly diverging from their ARMA(1,1) REE values.

These simulation results illustrate on the one hand the possibility of least squares learning converging to stationary solutions near the high inflation steady state. On the other hand these results also show that stability depends sensitively on the information available to agents when their inflation forecasts are made.

## 6.4 Learnable hyperinflations and stylized facts

A variety of stylized facts about hyperinflations have been noted in the literature. (Marcet and Nicolini 2003) list a low correlation between seigniorage revenues and inflation during a hyperinflation, recurrence of inflationary episodes within countries subject to them, and a high cross-country correlation between average inflation and seigniorage.

The most straightforward issue to consider within our model is the degree of correlation between seigniorage and inflation in the high-inflation equilibria of the type that can be stable under learning. Consider first the continuum of stationary "intrinsic nonfundamental equilibria" of the form (8). It is straightforward to compute the correlation between inflation $x_t$ and the seigniorage shock $u_t$:

$$Corr(x_t, u_t) = sgn(\gamma_0)\sqrt{\frac{1-\ell^2}{1+m^2+2\ell m}}, \text{ where } \ell = \frac{1-\beta_0}{\beta_1},\ m = \frac{\beta_0 - \gamma_0^{-1}}{\beta_1}.$$

Recalling that $\gamma_0$ can have any value, we can calculate numerically how the magnitude of $\gamma_0$ affects $Corr(x_t, u_t)$. As an illustration we consider a monthly model with the high inflation steady state given by $x^h = 4$. A monthly inflation rate of 300% is only slightly larger than the maximum inflation rates observed during various hyperinflations taking place in the 1980's in Argentina, Bolivia, and Peru, see (Marcet and Nicolini 2003). We set the elasticity of money demand with respect to inflation equal to $\beta_0 = -1$. The values for $\beta_1$ and $\alpha$ follow from equation (24) in Appendix A.1 and the steady state version of equation (4), respectively.

| $\gamma_0$ | $-0.5$ | $-0.2$ | $-0.1$ | $-0.05$ | 0 | 0.05 | 0.1 | 0.2 | 0.5 |
|---|---|---|---|---|---|---|---|---|---|
| $Corr$ | $-0.76$ | $-0.5$ | $-0.3$ | $-0.16$ | 0 | 0.18 | 0.36 | 0.65 | 0.96 |

Table 1: Correlation between seigniorage and inflation

As illustrated in Table 1, any value of $Corr(x_t, u_t)$ can occur, depending on which member of the set of learnable equilibria (8) obtains, as specified by the value of $\gamma_0$. Under learning, the value of $\gamma_0$ arises as an outcome of the learning process and is affected by the initial prior and the sequence of random shocks during the learning transition. For the more general class (5) of ARMA(1,1) solutions that also depend on uncorrelated sunspots $\eta_t'$, the magnitude of the correlation between $x_t$ and $u_t$ would be lower for each value of $\gamma_0$ than specified in the table.[12] Based on these results, it would therefore not be surprising to find that there is little systematic correlation between inflation and seigniorage in the data.

Recalling our various stability results, we see that there is the potential to have two distinct sets of learnable equilibria, namely $x^l$ and, if $\lambda$ is sufficiently low, the ARMA(1,1) solutions near $x^h$. This suggests that there may be ways to use these results to explain both recurrence of hyperinflations and high cross-country correlation between seigniorage and inflation. First, a higher mean level of seigniorage shrinks the domain of attraction of $x^l$ under steady state learning. This suggests the possibility that solutions near $x^h$ are more likely with high levels of mean seigniorage. Second, in our model we have taken $\lambda$ as given, but a natural extension would make $\lambda$ dependent on the level of inflation since higher inflation would provide an incentive to acquire contemporaneous information on inflation. Since a higher level of seigniorage increases $x^l$, this suggests an increased susceptibility to learning dynamics converging to the set of REE near $x^h$. To explain recurrence of periods of low

[12] Furthermore, simulations suggests that, for rational ARMA hyperinflation paths that start near $x^l$, the correlation between inflation and seigniorage is diminished during the transition.

and high inflation, one could invoke changes in policy at high rates of inflation as in (Marcet and Nicolini 2003). Alternatively, the reversion from high to low inflation might arise endogenously under constant gain learning.[13] The possibility of stable high inflation equilibria would yield different implications from the model of (Marcet and Nicolini 2003) if the policy regime remained unchanged. A more specific model of recurrent hyperinflations along these lines is left for future research.

# 7 Conclusions

In this paper we have studied the plausibility of stationary hyperinflation paths in the monetary inflation model by analyzing their stability under adaptive learning. The analysis has been conducted using a reduced form that has wider applicability. For the hyperinflation model, if agents can observe current endogenous variables at the time of forecasting then stationary hyperinflation paths of the AR(1) and ARMA(1,1) form, as well as associated sunspot solutions, are stable under learning. Although this suggests that these equilibria may provide a plausible explanation of hyperinflationary episodes, the finding is not robust to changes in agents' information set. In particular, if a significant share of agents cannot observe current endogenous variables when forming expectations, the stationary hyperinflation paths become unstable under learning.

On the other hand, the finding that the set of hyperinflationary equilibria is learnable when a sufficient number of agents have full contemporaneous information is new and - as suggested at the end of Section 6.4 - an extended version of the model could provide an alternative account for hyperinflations.

[13] The possibility of endogenously switching between two learnable equilibria under constant gain learning has been studied e.g. in Chapter 14 of (Evans and Honkapohja 2001) in a different model.

# A Appendices: Technical Details

## A.1 Linearization of hyperinflation model

Equation (3), which specifies the steady states, can be rewritten as

$$x^2 - \frac{\psi_1 + \psi_0 - g}{\psi_1} x + \frac{\psi_0}{\psi_1} = 0,$$

from which it follows that the two solutions $x^l < x^h$ satisfy $x^l x^h = \psi_0/\psi_1$ and hence that

$$x^l < \sqrt{\frac{\psi_0}{\psi_1}} < x^h.$$

Linearizing (2) at a steady state $x$ yields $x_t = \alpha + \beta_1 E_t^* x_{t+1} + \beta_0 E_{t-1}^* x_t + u_t$, where

$$\beta_0 = -\frac{\psi_1 x}{\psi_0 - \psi_1 x} \text{ and } \beta_1 = \frac{\psi_1 x^2}{\psi_0 - \psi_1 x}. \tag{24}$$

Note that $\beta_1 > 0$ and $\beta_0 < 0$. We remark that $-\beta_0$ is the elasticity of real money demand with respect to inflation and that $\beta_1 = -\beta_0 x$.

For the linearized model the AR(1) or ARMA(1,1) solutions of the form (5) are stationary if and only if the autoregressive parameter $\beta_1^{-1}(1-\beta_0) > 0$ is smaller than one. Since

$$\beta_1^{-1}(1 - \beta_0) = \frac{\psi_0}{\psi_1 x^2},$$

it follows that the solutions (5) are stationary near the high inflation steady state $x^h$, but explosive near the low inflation steady state $x^l$.

## A.2 Proof of Proposition 2

We start by defining $y_{t-1} = (1, x_{t-2})'$. With this notation we write the updating for the matrix of second moments as

$$R_t = R_{t-1} + \vartheta_t (y_{t-1} y_{t-1}' - R_{t-1})$$

and make a timing change $S_t = R_{t+1}$ in order to write recursive least squares (RLS) estimation as a stochastic recursive algorithm (SRA). In terms of $S_t$ we have

$$S_t = S_{t-1} + \vartheta_t \left( \frac{\vartheta_{t+1}}{\vartheta_t} \right) (y_t y_t' - S_{t-1}) \tag{25}$$

and

$$S_{t-1} = S_{t-2} + \vartheta_t(y_{t-1}y'_{t-1} - S_{t-2}) \tag{26}$$

for the periods $t$ and $t-1$. For updating of the estimates of the PLM parameters we have (17), which is rewritten in terms of $S_{t-1}$ as

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \vartheta_t S_{t-1}^{-1}(x_{t-1} - a_{t-1} - b_{t-1}x_{t-2}) \begin{pmatrix} 1 \\ x_{t-2} \end{pmatrix} \tag{27}$$

and

$$\begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} = \begin{pmatrix} a_{t-2} \\ b_{t-2} \end{pmatrix} + \vartheta_t \left(\frac{\vartheta_{t-1}}{\vartheta_t}\right) S_{t-2}^{-1}(x_{t-2} - a_{t-2} - b_{t-2}x_{t-3}) \begin{pmatrix} 1 \\ x_{t-3} \end{pmatrix}. \tag{28}$$

To write the entire system as a SRA we next define $\kappa_t = (a_t, b_t, a_{t-1}, b_{t-1})'$ and

$$\phi_t = \begin{pmatrix} \kappa_t \\ vecS_t \\ vecS_{t-1} \end{pmatrix} \text{ and } X_t = \begin{pmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ 1 \end{pmatrix}.$$

With this notation the equations for parameter updating are in the standard form

$$\phi_t = \phi_{t-1} + \vartheta_t Q(t, \phi_{t-1,} X_t), \tag{29}$$

where the function $Q(t, \phi_{t-1,} X_t)$ is defined by (25), (26), (27) and (28). We also write (16) in terms of general functional notation as

$$x_t = x_a(\phi_t) + x_b(\phi_t)x_{t-1} + x_u(\phi_t)u_t.$$

For the state vector $X_t$ we have

$$\begin{pmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ 1 \end{pmatrix} = \begin{pmatrix} x_b(\phi_{t-1}) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{t-2} \\ x_{t-3} \\ x_{t-4} \\ 1 \end{pmatrix} + \begin{pmatrix} x_a(\phi_{t-1}) & x_u(\phi_{t-1}) \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ u_{t-1} \end{pmatrix}$$

or

$$X_t = A(\phi_{t-1})X_{t-1} + B(\phi_{t-1})\upsilon_t, \tag{30}$$

where $\upsilon_t = (1, u_{t-1})'$.

The system (29) and (30) is a standard form for SRAs. Chapters 6 and 7 of (Evans and Honkapohja 2001) discuss the techniques for analyzing the convergence of SRAs. The convergence points and the conditions for convergence of dynamics generated by SRAs can be analyzed in terms of an associated ordinary differential equation (ODE). The SRA dynamics converge to an equilibrium point $\phi^*$ when $\phi^*$ is locally asymptotically fixed point of the associated differential equation. We now derive the associated ODE for our model.

For a fixed value of $\phi$ the state dynamics are essentially driven by the equation

$$x_{t-1}(\phi) = x_a(\phi) + x_b(\phi)x_{t-2} + x_u(\phi)u_{t-1}.$$

Now

$$Ey_t y_t' = \begin{pmatrix} 1 & Ex_t(\phi) \\ Ex_t(\phi) & Ex_t(\phi)^2 \end{pmatrix} \equiv M(\phi).$$

Defining $\epsilon_{t-1}(\phi) = x_{t-1} - a - bx_{t-2}$ we compute

$$\epsilon_{t-1}(\phi) = (x_a(\phi) - a) + (x_b(\phi) - b)x_{t-2}(\phi) + x_u(\phi)\upsilon_t,$$

so that

$$E\epsilon_{t-1}(\phi)\begin{pmatrix} 1 \\ x_{t-2}(\phi) \end{pmatrix} = M(\phi)\begin{pmatrix} x_a(\phi) - a \\ x_b(\phi) - b \end{pmatrix}.$$

These results yield the associated ODE as

$$\frac{d}{d\tau}\begin{pmatrix} a \\ b \end{pmatrix} = S^{-1}M(\phi)\begin{pmatrix} x_a(\phi) - a \\ x_b(\phi) - b \end{pmatrix}$$
$$\frac{dS}{d\tau} = M(\phi) - S$$

$$\frac{d}{d\tau}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = S_1^{-1}M(\phi)\begin{pmatrix} x_a(\phi) - a_1 \\ x_b(\phi) - b_1 \end{pmatrix}$$
$$\frac{dS_1}{d\tau} = M(\phi) - S_1,$$

where the temporary notation of variables with/without the subscript 1 refers to the $t$ and $t-1$ dating in the system (25), (26), (27) and (28).

A variant of the standard argument shows that stability of the ODE is controlled by the stability of the small ODE

$$\frac{d}{d\tau}\begin{pmatrix} a \\ b \\ a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} x_a(\phi) - a \\ x_b(\phi) - b \\ x_a(\phi) - a_1 \\ x_b(\phi) - b_1 \end{pmatrix}. \tag{31}$$

Next we linearize the small ODE at the fixed point $a = a_1 = a^* \equiv -\beta_1^{-1}\alpha$, $b = b_1 = b^* \equiv \beta_1^{-1}(1-\beta_0)$. The derivative of (31) at the fixed point can be written as $DX - I$, where

$$DX = \begin{pmatrix} \beta_0^{-1}\beta_1 & -\beta_0^{-1}\beta_1 & 1 & 0 \\ 0 & \beta_0^{-1} - 1 & 0 & 1 \\ \beta_0^{-1}\beta_1 & -\beta_0^{-1}\beta_1 & 1 & 0 \\ 0 & \beta_0^{-1} - 1 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $DX$ are clearly zero and the remaining two roots are $1 + \beta_0^{-1}\beta_1$ and $\beta_0^{-1}$. The local stability condition for the small ODE and hence the condition for local convergence the RLS learning as given in the statement of Proposition 2.

## A.3 Details on the Model with a Mixture of Agents

### A.3.1 The ALM when some agents do not observe current states

The coefficients in the ALM (18) are

$$A' = \begin{pmatrix} \varsigma(\alpha/\beta_1 + (1+\rho)[\lambda(1+b_1)a_1 + (1-\lambda)a_2]) & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \varsigma B_{11} & \varsigma B_{12} & \varsigma B_{13} & \varsigma B_{14} & \varsigma B_{15} & \varsigma B_{16} \\ \varsigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varsigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varsigma & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} \varsigma(\lambda(b_1 e_1 + c_1) + (1-\lambda)c_2 + 1/\beta_1) & \varsigma(\lambda(b_1 f_1 + d_1) + (1-\lambda)d_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where $\varsigma = (1/\beta_1 - (1-\lambda)b_2)^{-1}$ and

$$\begin{aligned}
B_{11} &= \lambda b_1^2 + \rho(1-\lambda)b_2 \\
B_{12} &= \rho\lambda b_1^2 \\
B_{13} &= \lambda b_1 c_1 + \rho(\lambda(b_1 e_1 + c_1) + (1-\lambda)c_2) \\
B_{14} &= \rho\lambda b_1 c_1 \\
B_{15} &= \lambda b_1 d_1 + \rho(\lambda(b_1 f_1 + d_1) + (1-\lambda)d_2) \\
B_{16} &= \rho\lambda b_1 d_1
\end{aligned}$$

### A.3.2 Eigenvalues in the small deficit case

For the hyperinflation model with a mixture of agents, the eigenvalues of the derivative of the T-map at the ARMA(1,1)-solution near the high-inflation steady state, for small deficit values (i.e. as $\xi \to 0$), are given by

$$\begin{aligned}
s_1 &= \frac{\lambda^2}{(1-\lambda)^2} \\
s_2 &= \frac{(1+\rho)(-1+\rho\lambda)}{\rho(-1+\lambda+\rho\lambda)} \\
s_3 &= -\frac{2\rho^3(-1+\lambda)\lambda^2 + \rho^5(\lambda^2 - 2\lambda^3) + \sqrt{s}}{2\rho^3(-1+\lambda)^2(1+(-1+\rho^2)\lambda)} \\
s_4 &= \frac{2\rho^3(-1+\lambda)\lambda^2 + \rho^5(\lambda^2 - 2\lambda^3) + \sqrt{s}}{2\rho^3(-1+\lambda)^2(1+(-1+\rho^2)\lambda)} \\
s_5 &= s_6 = 1 \\
s_7 &= s_8 = s_9 = s_{10} = 0
\end{aligned}$$

where

$$s = \rho^6\lambda^2(4(-1+\lambda)^2 + \rho^4\lambda(-4+5\lambda) - 4\rho^2(1-3\lambda+2\lambda^2))$$

and $\rho = -\omega$.

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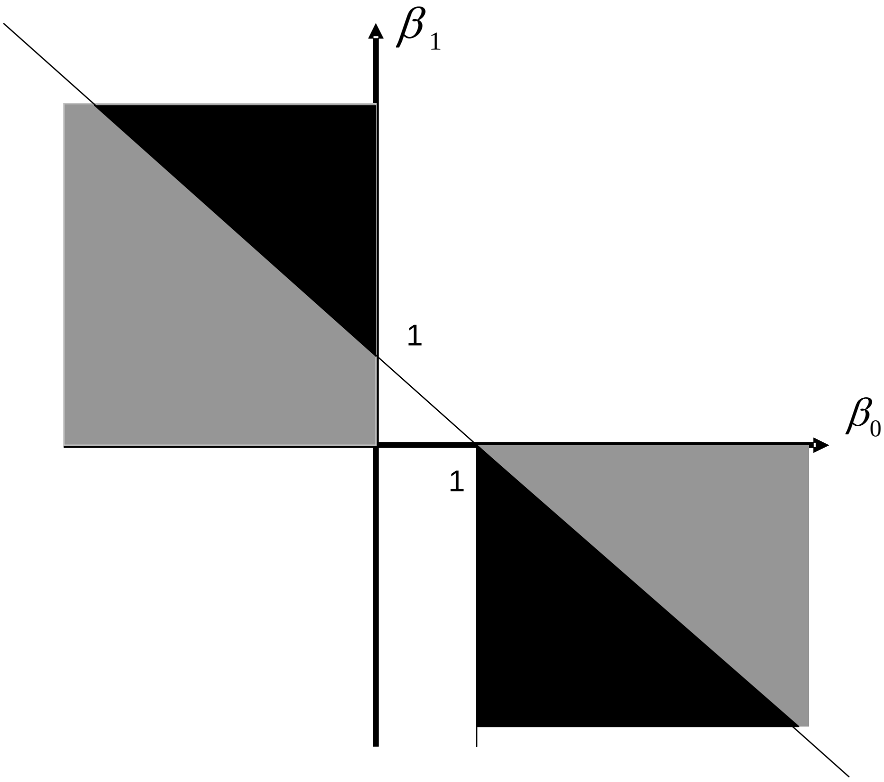


Figure 1: Regions of E-Stable ARMA equilibria

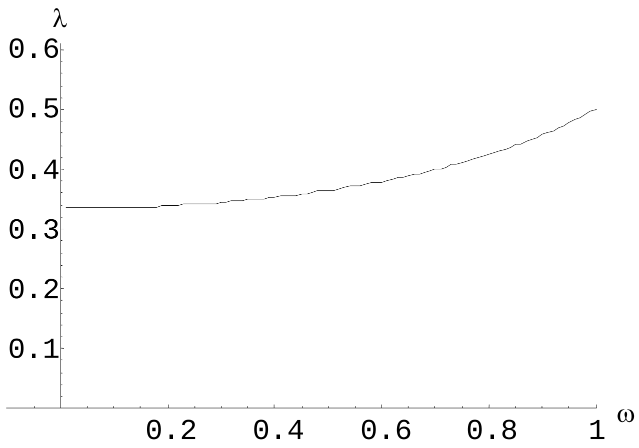


Figure 2: Critical value of $\lambda$, small deficit case ($\xi \rightarrow 0$)

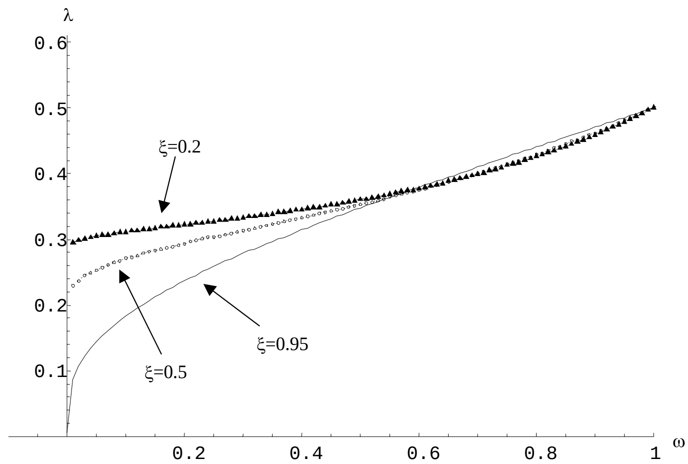


Figure 3: Critical value of $\lambda$, intermediate and large deficit case

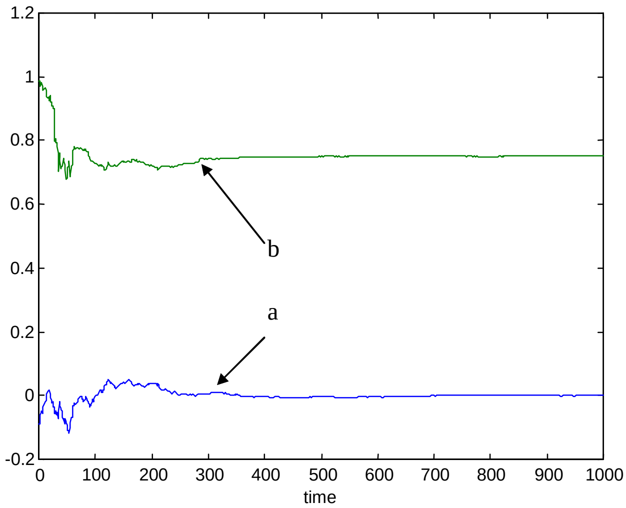


Figure 4: Example of convergence when $\lambda = 0$

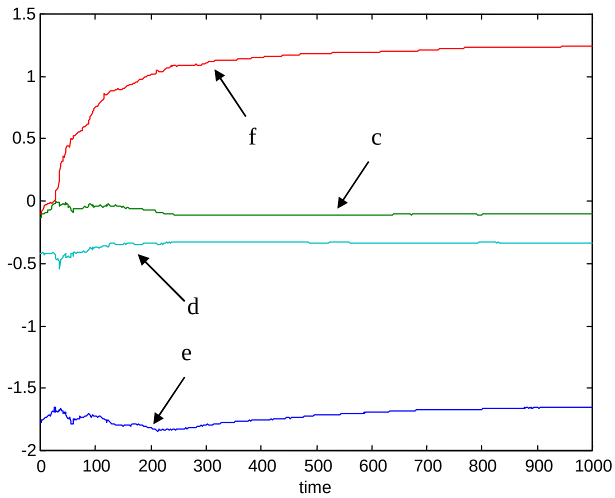


Figure 5: Example of convergence when $\lambda = 0$

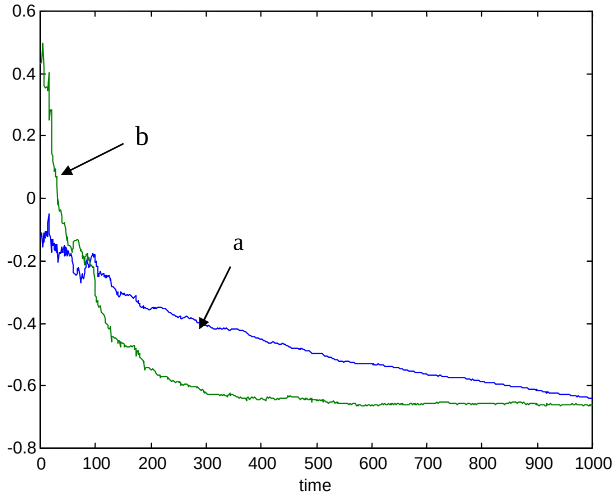


Figure 6: Example of divergence when $\lambda = 0.5$

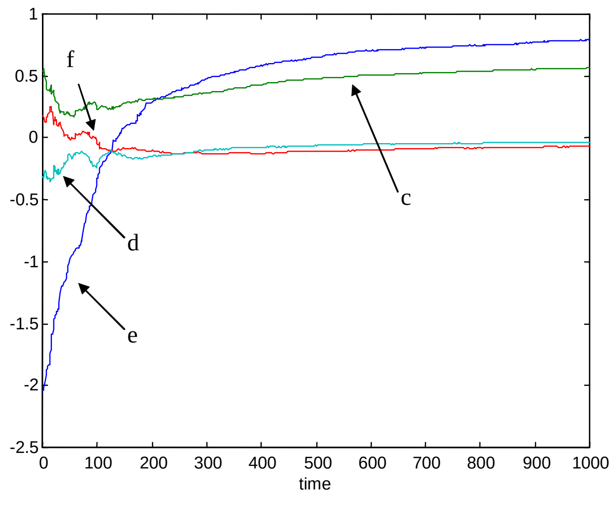


Figure 7: Example of divergence when $\lambda = 0.5$