

# Linearizing the Hyperinflation Model

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## Abstract

This is a technical appendix to "Are Stationary Hyperinflation Paths Learnable?" by the same authors which derives the linearization coefficients for the hyperinflation model at the high steady state.

## 1 Introduction

Consider the hyperinflation model with linear asset demand

$$m_t^d = \psi_0 - \psi_1 E_t \Pi_{t+1} \quad \text{with } \psi_0 > \psi_1 > 0$$

and money supply

$$m_t^s = \frac{m_{t-1}}{\Pi_t} + g + v_t$$

Setting  $m_t^s = m_t^d$  and imposing market clearing in  $t - 1$ , one obtains

$$\psi_0 - \psi_1 E_t \Pi_{t+1} = \frac{\psi_0 - \psi_1 E_{t-1} \Pi_t}{\Pi_t} + g + v_t$$

which implies

$$\Pi_t = \frac{\psi_0 - \psi_1 E_{t-1} \Pi_t}{\psi_0 - \psi_1 E_t \Pi_{t+1} - g - v_t} \quad (1)$$

Below we linearize this model around the high inflation steady state and determine the linearization coefficients  $\alpha$ ,  $\beta_1$ , and  $\beta_0$  of

$$x_t = \alpha + \beta_1 E_t x_{t+1} + \beta_0 E_{t-1} x_t + u_t$$

## 2 Steady states

The non-stochastic steady states of (1) satisfy

$$\Pi(\psi_0 - \psi_1 \Pi - g) - (\psi_0 - \psi_1 \Pi) = -\psi_1 \Pi^2 + (\psi_0 - g + \psi_1) \Pi - \psi_0 = 0$$

There are two steady states

$$\begin{aligned}\Pi^l &= -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \\ \Pi^h &= -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)\end{aligned}$$

where the high inflation steady state  $\Pi^h$  is the one around which we linearize.

### 3 Calculating the linearization coefficients at $\Pi^h$

$$\Pi^h = -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)$$

#### 3.1 The parameter $\alpha$ :

$$\begin{aligned}\alpha &= \frac{\psi_0 - \psi_1\Pi}{\psi_0 - \psi_1\Pi - g} \\ &= \frac{\psi_0 - \psi_1 \left( -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \right)}{\psi_0 - \psi_1 \left( -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \right) - g} \\ &= \frac{\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}{\psi_0 - g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}\end{aligned}\quad (2)$$

#### 3.2 The parameter $\beta_0$ :

$$\begin{aligned}\beta_0 &= -\frac{\psi_1}{\psi_0 - \psi_1\Pi - g} \\ &= -\frac{\psi_1}{\psi_0 - \psi_1 \left( -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \right) - g} \\ &= \frac{2\psi_1}{-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}\end{aligned}\quad (3)$$

#### 3.3 The parameter $\beta_1$ :

$$\beta_1 = \frac{(\psi_0 - \psi_1\Pi) \psi_1}{(\psi_0 - \psi_1\Pi - g)^2}$$

The denominator :

$$\begin{aligned}
& (\psi_0 - \psi_1 \Pi - g)^2 \\
&= (\psi_0 - \psi_1 \left( -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \right) - g)^2 \\
&= \frac{1}{4} \left( -\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)^2
\end{aligned}$$

The numerator:

$$\begin{aligned}
& (\psi_0 - \psi_1 \Pi) \psi_1 \\
&= \left( \psi_0 - \psi_1 \left( -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \right) \right) \psi_1 \\
&= \frac{1}{2} \left( \psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \psi_1
\end{aligned}$$

As a result

$$\begin{aligned}
\beta_1 &= \frac{(\psi_0 - \psi_1 \Pi) \psi_1}{(\psi_0 - \psi_1 \Pi - g)^2} \\
&= \frac{\frac{1}{2} \left( \psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \psi_1}{\frac{1}{4} \left( -\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)^2} \\
&= \frac{2 \left( \psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \psi_1}{\left( -\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)^2} \quad (4)
\end{aligned}$$

#### 4 Express coefficients in terms of $\omega = \frac{\psi_1}{\psi_0}$ and $\xi = \frac{g}{g_{\max}}$

Now, rewrite the linearization coefficients by dividing numerator and denominator by  $\psi_0$  :

$$\begin{aligned}
\alpha &= \frac{\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}{\psi_0 - g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} \\
&= \frac{\left( 1 + \frac{g}{\psi_0} - \frac{\psi_1}{\psi_0} - \sqrt{\left( 1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2 \right)} \right)}{\left( 1 - \frac{g}{\psi_0} - \frac{\psi_1}{\psi_0} - \sqrt{\left( 1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2 \right)} \right)}
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= \frac{2 \left( \psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \psi_1}{\left( -\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)^2} \\
&= \frac{2 \frac{\psi_1}{\psi_0} \left( 1 + \frac{g}{\psi_0} - \frac{\psi_1}{\psi_0} - \sqrt{\left( 1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2 \right)} \right)}{\left( -1 + \frac{g}{\psi_0} + \frac{\psi_1}{\psi_0} + \sqrt{\left( 1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2 \right)} \right)^2} \\
r &= \frac{\beta_0}{\beta_1} = \frac{-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}{\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} \\
&= \frac{\left( -1 + \frac{g}{\psi_0} + \frac{\psi_1}{\psi_0} + \sqrt{\left( 1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2 \right)} \right)}{\left( 1 + \frac{g}{\psi_0} - \frac{\psi_1}{\psi_0} - \sqrt{\left( 1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2 \right)} \right)}
\end{aligned}$$

The coefficients now depend on

$$\omega = \psi_1/\psi_0$$

and

$$g/\psi_0$$

only. As argued in the text, the maximum deficit is

$$g^{\max} = \left( \sqrt{\psi_1} - \sqrt{\psi_0} \right)^2$$

Defining

$$\xi = g/g^{\max}$$

we have

$$\frac{g}{\psi_0} = \xi \left( 1 + \frac{\psi_1}{\psi_0} - 2\sqrt{\frac{\psi_1}{\psi_0}} \right)$$

Using this, we can express the coefficients as functions of  $\xi$  and  $\omega$ , which is the result used in the Mathematica Notebook:

$$\alpha = \frac{\left( 1 + \xi(1 + \omega - 2\sqrt{\omega}) - \omega - \sqrt{\left( 1 - 2\xi(1 + \omega - 2\sqrt{\omega}) - 2\omega + (\xi(1 + \omega - 2\sqrt{\omega}))^2 - 2\xi(1 + \omega - 2\sqrt{\omega}) + (\omega)^2 \right)} \right)}{\left( 1 - \xi(1 + \omega - 2\sqrt{\omega}) - \omega - \sqrt{\left( 1 - 2\xi(1 + \omega - 2\sqrt{\omega}) - 2\omega + (\xi(1 + \omega - 2\sqrt{\omega}))^2 - 2\xi(1 + \omega - 2\sqrt{\omega}) + (\omega)^2 \right)} \right)}$$

$$\beta_1 = \frac{2\omega \left( 1 + \xi(1 + \omega - 2\sqrt{\omega}) - \omega - \sqrt{(1 - 2\xi(1 + \omega - 2\sqrt{\omega}) - 2\omega + (\xi(1 + \omega - 2\sqrt{\omega}))^2 - 2\omega\xi(1 + \omega - 2\sqrt{\omega}) + (\omega)^2)} \right)}{\left( -1 + \xi(1 + \omega - 2\sqrt{\omega}) + \omega + \sqrt{(1 - 2\xi(1 + \omega - 2\sqrt{\omega}) - 2\omega + (\xi(1 + \omega - 2\sqrt{\omega}))^2 - 2\omega\xi(1 + \omega - 2\sqrt{\omega}) + (\omega)^2)} \right)^2}$$

$$r = \frac{\left( -1 + \xi(1 + \omega - 2\sqrt{\omega}) + \omega + \sqrt{(1 - 2\xi(1 + \omega - 2\sqrt{\omega}) - 2\omega + (\xi(1 + \omega - 2\sqrt{\omega}))^2 - 2\omega\xi(1 + \omega - 2\sqrt{\omega}) + (\omega)^2)} \right)}{\left( 1 + \xi(1 + \omega - 2\sqrt{\omega}) - \omega - \sqrt{(1 - 2\xi(1 + \omega - 2\sqrt{\omega}) - 2\omega + (\xi(1 + \omega - 2\sqrt{\omega}))^2 - 2\omega\xi(1 + \omega - 2\sqrt{\omega}) + (\omega)^2)} \right)}$$

## 5 The linearization for $g \rightarrow 0$

### 5.1 The constant term $\alpha$ :

From equation (2):

$$\alpha = \frac{\psi_0 - \psi_1 \Pi}{\psi_0 - \psi_1 \Pi - d}$$

$$= -\frac{\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}{-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} \quad (5)$$

As  $g \rightarrow 0$  numerator and denominator converge to zero. Therefore, we must apply De L'Hopital.

The derivative of the numerator:

$$\frac{\partial \left( \psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)}{\partial g}$$

$$= 1 - \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} (-2\psi_0 + 2g - 2\psi_1)$$

Set  $g = 0$  and evaluate this expression using  $\psi_0 > \psi_1$ :

$$1 - \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} (-2\psi_0 + 2g - 2\psi_1)$$

$$= 1 - \frac{1}{2\sqrt{((- \psi_0 + \psi_1)^2)}} (-2\psi_1 - 2\psi_0) = 2\frac{\psi_0}{\psi_0 - \psi_1}$$

The derivative of the denominator:

$$\frac{\partial \left( -\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)}{\partial g}$$

$$= 1 + \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} (-2\psi_0 + 2g - 2\psi_1)$$

Set  $g = 0$  and evaluate using  $\psi_0 > \psi_1$ :

$$\begin{aligned} & 1 + \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} (-2\psi_0 + 2g - 2\psi_1) \\ &= 1 + \frac{1}{2\sqrt{((-\psi_0 + \psi_1)^2)}} (-2\psi_1 - 2\psi_0) = -2\frac{\psi_1}{\psi_0 - \psi_1} \end{aligned}$$

As a result, the constant converges for  $g \rightarrow 0$  to

$$\begin{aligned} \lim_{g \rightarrow 0} \alpha &= \lim_{g \rightarrow 0} \frac{\psi_0 - \psi_1 \Pi^h(g)}{\psi_0 - \psi_1 \Pi^h(g) - g} \\ &= -\frac{2\frac{\psi_0}{\psi_0 - \psi_1}}{-2\frac{\psi_1}{\psi_0 - \psi_1}} = \frac{\psi_0}{\psi_1} \end{aligned}$$

## 5.2 The parameter $\beta_0$ on $E_{t-1}\Pi_t$ :

We have

$$\beta_0 = -\frac{\psi_1}{\psi_0 - \psi_1 \Pi - g}$$

from equation (3). Since money is valued, the denominator is larger than zero. Therefore,

$$\lim_{g \rightarrow 0} \beta_0 = -\infty$$

## 5.3 The parameter $\beta_1$ on $E_t\Pi_{t+1}$ :

From (4) we have

$$\beta_1 = \frac{(\psi_0 - \psi_1 \Pi) \psi_1}{(\psi_0 - \psi_1 \Pi - g)^2}$$

Since both denominator and numerator converge to zero, we must apply De L'Hopital again.

From the calculations following (5), we know I know that the derivative of the numerator converges to

$$2\frac{\psi_0\psi_1}{\psi_0 - \psi_1}$$

The denominator is given by

$$\begin{aligned}
& (\psi_0 - \psi_1 \Pi - g)^2 \\
&= (\psi_0 - \psi_1 \left( -\frac{1}{2\psi_1} \left( -\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \right) - g)^2 \\
&= \frac{1}{4} \left( -\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)^2
\end{aligned}$$

Taking the derivative w.r.t.  $g$  delivers

$$\begin{aligned}
& \frac{\partial \left( \frac{1}{4} \left( -\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right)^2 \right)}{\partial g} = \\
&= \frac{1}{2} \left( -\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)} \right) \cdot \\
& \left( 1 + \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1 \psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} (-2\psi_0 + 2g - 2\psi_1) \right)
\end{aligned}$$

Setting  $g = 0$  and evaluating this expression delivers that it is equal to zero.

This implies that

$$\lim_{g \rightarrow 0} \beta_1 = \frac{2 \frac{\psi_0 \psi_1}{\psi_0 - \psi_1}}{0} = +\infty$$

#### 5.4 The ratio $r = \frac{\beta_0}{\beta_1}$ as $g \rightarrow 0$

The limit for  $r = \frac{\beta_0}{\beta_1}$  is given by

$$\begin{aligned}
\lim_{g \rightarrow 0} r &= \lim_{g \rightarrow 0} \frac{\beta_0}{\beta_1} \\
&= \lim_{g \rightarrow 0} \frac{-\frac{\psi_1}{\psi_0 - \psi_1 \Pi - g}}{\frac{(\psi_0 - \psi_1 \Pi) \psi_1}{(\psi_0 - \psi_1 \Pi - g)^2}} \\
&= \lim_{g \rightarrow 0} \frac{-\psi_0 + \psi_1 \Pi + g}{\psi_0 - \psi_1 \Pi} \\
&= \lim_{g \rightarrow 0} -\frac{\psi_0 - \psi_1 \Pi - g}{\psi_0 - \psi_1 \Pi} \\
&= -\frac{\psi_1}{\psi_0}
\end{aligned}$$

where the last line follows from the calculations in section 5.1.