Linearizing the Hyperinflation Model

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Abstract

This is a technical appendix to "Are Stationary Hyperinflation Paths Learnable?" by the same authors which derives the linearization coefficients for the hyperinflation model at the high steady state.

1 Introduction

Consider the hyperinflation model with linear asset demand

$$m_t^d = \psi_0 - \psi_1 E_t \Pi_{t+1} \qquad \omega ith\ \psi_0 > \psi_1 > 0$$

and money supply

$$m_t^s = \frac{m_{t-1}}{\Pi_t} + g + v_t$$

Setting $m_t^s = m_t^d$ and imposing market clearing in $t-1$, one obtains

$$\psi_0 - \psi_1 E_t \Pi_{t+1} = \frac{\psi_0 - \psi_1 E_{t-1} \Pi_t}{\Pi_t} + g + v_t$$

which implies

$$\Pi_t = \frac{\psi_0 - \psi_1 E_{t-1} \Pi_t}{\psi_0 - \psi_1 E_t \Pi_{t+1} - g - v_t} \tag{1}$$

Below we linearize this model around the high inflation steady state and determine the linearization coefficients α, β_1, and β_0 of

$$x_t = \alpha + \beta_1 E_t x_{t+1} + \beta_0 E_{t-1} x_t + u_t$$

2 Steady states

The non-stochastic steady states of (1) satisfy

$$\Pi(\psi_0 - \psi_1 \Pi - g) - (\psi_0 - \psi_1 \Pi) = -\psi_1 \Pi^2 + (\psi_0 - g + \psi_1)\Pi - \psi_0 = 0$$

The are two steady states

$$\Pi^l = -\frac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)$$
$$\Pi^h = -\frac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)$$

where the high inflation steady state Π^h is the one around which we linearize.

3 Calculating the linearization coefficients at Π^h

$$\Pi^h = -\tfrac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)$$

3.1 The parameter α:

$$\begin{aligned}
\alpha &= \frac{\psi_0 - \psi_1\Pi}{\psi_0 - \psi_1\Pi - g} \\
&= \frac{\psi_0 - \psi_1\left(-\tfrac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\right)}{\psi_0 - \psi_1\left(-\tfrac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\right) - g} \\
&= \frac{\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}{\psi_0 - g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} \qquad (2)
\end{aligned}$$

3.2 The parameter β_0:

$$\begin{aligned}
\beta_0 &= -\frac{\psi_1}{\psi_0 - \psi_1\Pi - g} \\
&= -\frac{\psi_1}{\psi_0 - \psi_1(-\tfrac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)) - g} \\
&= \frac{2\psi_1}{-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} \qquad (3)
\end{aligned}$$

3.3 The parameter β_1:

$$\beta_1 = \frac{(\psi_0 - \psi_1\Pi)\,\psi_1}{(\psi_0 - \psi_1\Pi - g)^2}$$

The denominator :

$$
\begin{aligned}
&(\psi_0 - \psi_1\Pi - g)^2 \\
&= (\psi_0 - \psi_1\left(-\frac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\right) - g)^2 \\
&= \frac{1}{4}\left(-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)^2
\end{aligned}
$$

The numerator:

$$
\begin{aligned}
&(\psi_0 - \psi_1\Pi)\,\psi_1 \\
&= \left(\psi_0 - \psi_1\left(-\frac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\right)\right)\psi_1 \\
&= \frac{1}{2}\left(\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\psi_1
\end{aligned}
$$

As a result

$$
\begin{aligned}
\beta_1 &= \frac{(\psi_0 - \psi_1\Pi)\,\psi_1}{(\psi_0 - \psi_1\Pi - g)^2} \\
&= \frac{\frac{1}{2}\left(\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\psi_1}{\frac{1}{4}\left(-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)^2} \\
&= \frac{2\left(\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\psi_1}{\left(-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)^2} \qquad (4)
\end{aligned}
$$

4 Express coefficients in terms of $\omega = \frac{\psi_1}{\psi_0}$ and $\xi = \frac{g}{g^{\max}}$

Now, rewrite the linearization coefficients by dividing numerator and denominator by ψ_0 :

$$
\begin{aligned}
\alpha &= \frac{\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}{\psi_0 - g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} \\
&= \frac{\left(1 + \frac{g}{\psi_0} - \frac{\psi_1}{\psi_0} - \sqrt{\left(1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2\right)}\right)}{\left(1 - \frac{g}{\psi_0} - \frac{\psi_1}{\psi_0} - \sqrt{\left(1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2\right)}\right)}
\end{aligned}
$$

$$\beta_1 = \frac{2\left(\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\psi_1}{\left(-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)^2}$$

$$= \frac{2\frac{\psi_1}{\psi_0}\left(1 + \frac{g}{\psi_0} - \frac{\psi_1}{\psi_0} - \sqrt{\left(1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2\right)}\right)}{\left(-1 + \frac{g}{\psi_0} + \frac{\psi_1}{\psi_0} + \sqrt{\left(1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2\right)}\right)^2}$$

$$r = \frac{\beta_0}{\beta_1} = \frac{-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}{\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}$$

$$= \frac{\left(-1 + \frac{g}{\psi_0} + \frac{\psi_1}{\psi_0} + \sqrt{\left(1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2\right)}\right)}{\left(1 + \frac{g}{\psi_0} - \frac{\psi_1}{\psi_0} - \sqrt{\left(1 - 2\frac{g}{\psi_0} - 2\frac{\psi_1}{\psi_0} + \left(\frac{g}{\psi_0}\right)^2 - \frac{2g\psi_1}{\psi_0^2} + \left(\frac{\psi_1}{\psi_0}\right)^2\right)}\right)}$$

The coefficients now depend on

$$\omega = \psi_1/\psi_0$$

and

$$g/\psi_0$$

only. As argued in the text, the maximum deficit is

$$g^{\max} = \left(\sqrt{\psi_1} - \sqrt{\psi_0}\right)^2$$

Defining

$$\xi = g/g^{\max}$$

we have

$$\frac{g}{\psi_0} = \xi(1 + \frac{\psi_1}{\psi_0} - 2\sqrt{\frac{\psi_1}{\psi_0}})$$

Using this, we can express the coefficients as functions of ξ and ω, which is the result used in the Mathematica Notebook:

$$\alpha = \frac{\left(1+\xi(1+\omega-2\sqrt{\omega})-\omega-\sqrt{\left(1-2\xi(1+\omega-2\sqrt{\omega})-2\omega+\left(\xi(1+\omega-2\sqrt{\omega})\right)^2-2\omega\xi(1+\omega-2\sqrt{\omega})+(\omega)^2\right)}\right)}{\left(1-\xi(1+\omega-2\sqrt{\omega})-\omega-\sqrt{\left(1-2\xi(1+\omega-2\sqrt{\omega})-2\omega+\left(\xi(1+\omega-2\sqrt{\omega})\right)^2-2\omega\xi(1+\omega-2\sqrt{\omega})+(\omega)^2\right)}\right)}$$

$$\beta_1 = \frac{2\omega\left(1+\xi(1+\omega-2\sqrt{\omega})-\omega-\sqrt{\left(1-2\xi(1+\omega-2\sqrt{\omega})-2\omega+\left(\xi(1+\omega-2\sqrt{\omega})\right)^2-2\omega\xi(1+\omega-2\sqrt{\omega})+(\omega)^2\right)}\right)}{\left(-1+\xi(1+\omega-2\sqrt{\omega})+\omega+\sqrt{\left(1-2\xi(1+\omega-2\sqrt{\omega})-2\omega+\left(\xi(1+\omega-2\sqrt{\omega})\right)^2-2\omega\xi(1+\omega-2\sqrt{\omega})+(\omega)^2\right)}\right)^2}$$

$$r = \frac{\left(-1+\xi(1+\omega-2\sqrt{\omega})+\omega+\sqrt{\left(1-2\xi(1+\omega-2\sqrt{\omega})-2\omega+\left(\xi(1+\omega-2\sqrt{\omega})\right)^2-2\omega\xi(1+\omega-2\sqrt{\omega})+(\omega)^2\right)}\right)}{\left(1+\xi(1+\omega-2\sqrt{\omega})-\omega-\sqrt{\left(1-2\xi(1+\omega-2\sqrt{\omega})-2\omega+\left(\xi(1+\omega-2\sqrt{\omega})\right)^2-2\omega\xi(1+\omega-2\sqrt{\omega})+(\omega)^2\right)}\right)}$$

5 The linearization for $g \to 0$

5.1 The constant term α:

From equation (2):

$$\begin{aligned}
\alpha &= \frac{\psi_0 - \psi_1 \Pi}{\psi_0 - \psi_1 \Pi - d} \\
&= -\frac{\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}{-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}} \qquad (5)
\end{aligned}$$

As $g \to 0$ numerator and denominator converge to zero. Therefore, we must apply De L'Hopital.

The derivative of the numerator:

$$\begin{aligned}
&\frac{\partial\left(\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)}{\partial g} \\
&= 1 - \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}\left(-2\psi_0 + 2g - 2\psi_1\right)
\end{aligned}$$

Set $g = 0$ and evaluate this expression using $\psi_0 > \psi_1$:

$$\begin{aligned}
&1 - \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}\left(-2\psi_0 + 2g - 2\psi_1\right) \\
&= 1 - \frac{1}{2\sqrt{\left((-\psi_0 + \psi_1)^2\right)}}\left(-2\psi_1 - 2\psi_0\right) = 2\frac{\psi_0}{\psi_0 - \psi_1}
\end{aligned}$$

The derivative of the denominator:

$$\begin{aligned}
&\frac{\partial\left(-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)}{\partial g} \\
&= 1 + \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}\left(-2\psi_0 + 2g - 2\psi_1\right)
\end{aligned}$$

Set $g = 0$ and evaluate using $\psi_0 > \psi_1$:

$$1 + \frac{1}{2\sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}}(-2\psi_0 + 2g - 2\psi_1)$$
$$= 1 + \frac{1}{2\sqrt{\left((-\psi_0 + \psi_1)^2\right)}}(-2\psi_1 - 2\psi_0) = -2\frac{\psi_1}{\psi_0 - \psi_1}$$

As a result, the constant converges for $g \to 0$ to

$$\lim_{g\to 0} \alpha = \lim_{g\to 0} \frac{\psi_0 - \psi_1 \Pi^h(g)}{\psi_0 - \psi_1 \Pi^h(g) - g}$$
$$= -\frac{2\frac{\psi_0}{\psi_0 - \psi_1}}{-2\frac{\psi_1}{\psi_0 - \psi_1}} = \frac{\psi_0}{\psi_1}$$

5.2 The parameter β_0 on $E_{t-1}\Pi_t$:

We have

$$\beta_0 = -\frac{\psi_1}{\psi_0 - \psi_1 \Pi - g}$$

from equation (3). Since money is valued, the denominator is larger than zero. Therefore,

$$\lim_{g\to 0} \beta_0 = -\infty$$

5.3 The parameter β_1 on $E_t\Pi_{t+1}$:

From (4) we have

$$\beta_1 = \frac{(\psi_0 - \psi_1 \Pi)\,\psi_1}{(\psi_0 - \psi_1 \Pi - g)^2}$$

Since both denominator and numerator converge to zero, we must apply De L'Hopital again.

From the calculations following (5), we know I know that the derivative of the numerator converges to

$$2\frac{\psi_0 \psi_1}{\psi_0 - \psi_1}$$

The denominator is given by

$$
\begin{aligned}
&(\psi_0 - \psi_1 \Pi - g)^2 \\
&= \left(\psi_0 - \psi_1 \left(-\frac{1}{2\psi_1}\left(-\psi_0 + g - \psi_1 - \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)\right) - g\right)^2 \\
&= \frac{1}{4}\left(-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right)^2
\end{aligned}
$$

Taking the derivative w.r.t. g delivers

$$
\begin{aligned}
&\frac{\partial\left(\frac{1}{4}\left(-\psi_0+g+\psi_1+\sqrt{(\psi_0^2-2\psi_0 g-2\psi_1\psi_0+g^2-2g\psi_1+\psi_1^2)}\right)^2\right)}{\partial g} = \\
&= \tfrac{1}{2}\left(-\psi_0 + g + \psi_1 + \sqrt{(\psi_0^2 - 2\psi_0 g - 2\psi_1\psi_0 + g^2 - 2g\psi_1 + \psi_1^2)}\right) \cdot \\
&\left(1 + \frac{1}{2\sqrt{(\psi_0^2-2\psi_0 g-2\psi_1\psi_0+g^2-2g\psi_1+\psi_1^2)}}\left(-2\psi_0 + 2g - 2\psi_1\right)\right)
\end{aligned}
$$

Setting $g = 0$ and evaluating this expression delivers that it is equal to zero. This implies that

$$
\lim_{g\to 0} \beta_1 = \frac{2\frac{\psi_0\psi_1}{\psi_0-\psi_1}}{0} = +\infty
$$

5.4 The ratio $r = \frac{\beta_0}{\beta_1}$ as $g \to 0$

The limit for $r = \frac{\beta_0}{\beta_1}$ is given by

$$
\begin{aligned}
\lim_{g\to 0} r &= \lim_{g\to 0} \frac{\beta_0}{\beta_1} \\
&= \lim_{g\to 0} \frac{-\frac{\psi_1}{\psi_0-\psi_1\Pi-g}}{\frac{(\psi_0-\psi_1\Pi)\psi_1}{(\psi_0-\psi_1\Pi-g)^2}} \\
&= \lim_{g\to 0} \frac{-\psi_0 + \psi_1\Pi + g}{\psi_0 - \psi_1\Pi} \\
&= \lim_{g\to 0} -\frac{\psi_0 - \psi_1\Pi - g}{\psi_0 - \psi_1\Pi} \\
&= -\frac{\psi_1}{\psi_0}
\end{aligned}
$$

where the last line follows from the calculations in section 5.1.