Online Appendix to "Optimal Trend Inflation"

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July 12, 2017

This appendix spells out details of the multi-sector model in sections 10 and 11 of the main text and presents proofs for the propositions and lemmas stated in these sections.

1 Decentralized Economy

1.1 Household and Government Budget Constraints

In the multi-sector model, the representative household has the same preferences as in the one-sector model, but faces the modified flow budget constraint

$$C_t + K_{t+1} + \frac{B_t}{P_t} = (r_t + 1 - d)K_t + \frac{W_t}{P_t}L_t + \sum_{z=1}^{Z}\left(\int_0^1 \frac{\Theta_{jzt}}{P_t}\,\mathrm{dj}\right) + \frac{B_{t-1}}{P_t}(1 + i_{t-1}) - T_t,$$

where Θ_{jzt} denotes nominal profits from ownership of firm j in sector $z = 1, \ldots Z$. The government faces the budget constraint

$$\frac{B_t}{P_t} = \frac{B_{t-1}}{P_t}(1 + i_{t-1}) + \sum_{z=1}^{Z}\tau\left(\int_0^1 \left(\frac{P_{jzt}}{P_t}\right) Y_{jzt}\,\mathrm{dj}\right) - T_t.$$

1.2 Sectoral Technology, Marginal Costs and Price Setting

Sectoral technology. Output Y_{zt} of sector z combines intermediate products $j \in [0, 1]$ according to

$$Y_{zt} = \left(\int_0^1 Y_{jzt}^{\frac{\theta-1}{\theta}}\, dj\right)^{\frac{\theta}{\theta-1}} \quad , \quad \theta > 1 \ .$$

Cost minimization yields $P_{zt} = \left(\int_0^1 P_{jzt}^{1-\theta}\, dj\right)^{\frac{1}{1-\theta}}$ and the usual demand functions. Firm j in sector z uses technology $Y_{jzt} = A_{zt}Q_{zt-s_{jzt}}G_{jzt}\left(K_{jzt}^{1-1/\phi}L_{jzt}^{1/\phi} - F_{zt}\right)$. The idiosyncratic δ-shock in sector z occurs at the rate $\delta_z \geq 0$.

Marginal costs. Firm j hires labor and capital at economy-wide and perfectly competitive factor markets. The cost minimization problem of firm j in sector z yields the first order conditions that imply that firms $j \in [0, 1]$ maintain the same optimal capital labor ratio in all sectors and implies that marginal costs correspond to

$$MC_t = \left(\frac{W_t}{1/\phi}\right)^{\frac{1}{\phi}} \left(\frac{P_t r_t}{1 - 1/\phi}\right)^{1-\frac{1}{\phi}} . \tag{1}$$

Price setting. Let P_{jzt} denote the price charged by firm j in sector z in period t. The firms in this sector that receive a δ-shock can freely choose the product price but otherwise can adjust prices only with probability $\alpha_z \in (0,1)$ in each period. Thus, firm j sets its optimal price by solving:

$$\max_{P_{jzt}} E_t \sum_{i=0}^{\infty} (\alpha_z(1-\delta_z))^i \frac{\Omega_{t,t+i}}{P_{t+i}} \left[(1+\tau_z)P_{jzt}Y_{jzt+i} - MC_{t+i}I_{jzt+i}\right] \tag{2}$$

$$s.t. \quad Y_{jzt+i} = A_{zt+i}Q_{zt+i}\mathcal{Q}_{jzt+i}\left(I_{jzt+i} - F_{zt+i}\right),$$

$$Y_{jzt+i} = \psi_z \left(\frac{P_{jzt}}{P_{zt+i}}\right)^{-\theta} \left(\frac{P_{zt+i}}{P_{t+i}}\right)^{-1} Y_{t+i}.$$

$I_{jzt} = F_{zt} + Y_{jzt}/(A_{zt}Q_{zt}\mathcal{Q}_{jzt})$ denotes the units of factor inputs $(K_{jzt}^{1-\frac{1}{\phi}} L_{jzt}^{\frac{1}{\phi}})$ required to produce Y_{jzt} units of output, $\mathcal{Q}_{jzt} = Q_{zt-s_{jzt}}G_{jzt}/Q_{zt}$, and $\Omega_{t,t+i}$ denotes the representative household's discount factor between periods t and $t+i$. The optimal price of firm j in sector z evolves according to

$$\frac{P^{\star}_{jzt}}{P_t}\mathcal{Q}_{jzt} = \left(\frac{1}{1+\tau}\frac{\theta}{\theta-1}\right)\frac{N_{zt}}{D_{zt}}, \tag{3}$$

$$D_{zt} = 1 + \alpha_z(1-\delta_z)E_t\left[\Omega_{t,t+1}\left(\frac{P_{zt+1}}{P_{zt}}\right)^{\theta-1}\left(\frac{Y_{t+1}}{Y_t}\right)D_{zt+1}\right], \tag{4}$$

$$N_{zt} = \frac{MC_t}{P_tA_{zt}Q_{zt}} + \alpha_z(1-\delta_z)E_t\left[\Omega_{t,t+1}\left(\frac{P_{zt+1}}{P_{zt}}\right)^{\theta-1}\left(\frac{P_{t+1}}{P_t}\right)\left(\frac{Y_{t+1}}{Y_t}\right)\left(\frac{q_{zt+1}}{g_{zt+1}}\right)N_{zt+1}\right]. \tag{5}$$

1.3 Aggregation and Market Clearing

Sectoral and aggregate price levels. We express the sectoral price level recursively following analogous steps to the steps that we use in the one sector model to derive the aggregate price level. This yields

$$P_{zt}^{1-\theta} = \left[\alpha_z\delta_z + (1-\alpha_z)(\Delta^e_{zt})^{1-\theta}\right](P^{\star}_{z,t,t})^{1-\theta} + \alpha_z(1-\delta_z)(P_{zt-1})^{1-\theta}. \tag{6}$$

Here, $P^{\star}_{z,t-s,t}$ ($P^{\star}_{z,t,t}$) denotes the optimal price of the firm that received a δ shock s (zero) periods ago and belongs to sector z. Δ^e_{zt} denotes the productivity adjustment factor in sector z in the efficient economy, which is derived below. This factor can be shown to evolve recursively as

$$(\Delta^e_{zt})^{1-\theta} = \delta_z + (1-\delta_z)\left(\Delta^e_{zt-1}q_{zt}/g_{zt}\right)^{1-\theta}. \tag{7}$$

Equation (6) implies that

$$1 = \left[\alpha_z\delta_z + (1-\alpha_z)(\Delta^e_{zt})^{1-\theta}\right](p^{\star}_{zt})^{1-\theta} + \alpha_z(1-\delta_z)\Pi_{zt}^{\theta-1},$$

using the definitions $p^\star_{zt} = P^\star_{z,t,t}/P_{zt}$ and $\Pi_{zt} = P_{zt}/P_{zt-1}$. Cobb-Douglas aggregation of sectoral output also implies that the aggregate price level corresponds to

$$P_t = \prod_{z=1}^{Z} \left(\frac{P_{zt}}{\psi_z}\right)^{\psi_z}.$$

Sectoral and aggregate technologies. We define the sectoral productivity factor as

$$\Delta_{zt} = \int_0^1 \left(\frac{1}{Q_{jzt}}\right)\left(\frac{P_{jzt}}{P_{zt}}\right)^{-\theta} dj.$$

Following corresponding steps as in the one sector economy, this equation can be expressed recursively according to

$$\Delta_{zt} = \left[\alpha_z\delta_z + (1-\alpha_z)(\Delta^e_{zt})^{1-\theta}\right](p^\star_{zt})^{-\theta} + \alpha_z(1-\delta_z)\left(\frac{q_{zt}}{g_{zt}}\right)\Pi^\theta_{zt}\Delta_{zt-1}. \tag{8}$$

It can also be shown that the sectoral technology corresponds to

$$Y_{zt} = \frac{A_{zt}Q_{zt}}{\Delta_{zt}}\left(K_{zt}^{1-\frac{1}{\phi}} L_{zt}^{\frac{1}{\phi}} - F_{zt}\right),$$

using the definitions $L_{zt} = \int_0^1 L_{jzt}$ dj and $K_{zt} = \int_0^1 K_{jzt}$ dj. Augmenting economy-wide labor market clearing $L_t = \sum_z L_{zt}$ according to

$$\left(\frac{K_t}{L_t}\right)^{1-\frac{1}{\phi}} L_t = \sum_{z=1}^{Z}\left(\frac{K_{zt}}{L_{zt}}\right)^{1-\frac{1}{\phi}} L_{zt}$$

and rewriting it using sectoral technology and $K_t = \sum_z K_{zt}$ yields

$$K_t^{1-\frac{1}{\phi}} L_t^{\frac{1}{\phi}} - F_t = \sum_z \left(Y_{zt}\frac{\Delta_{zt}}{A_{zt}Q_{zt}}\right),$$

with $F_t = \sum_z F_{zt}$. We then replace sectoral output by aggregate output using demand functions $Y_{zt} = \psi_z\,(P_{zt}/P_t)^{-1}\,Y_t$. This yields the aggregate technology

$$Y_t = \frac{(\Gamma^e_t)^{1/\phi}}{\Delta_t}\left(K_t^{1-\frac{1}{\phi}} L_t^{\frac{1}{\phi}} - F_t\right), \tag{9}$$

denoting the aggregate productivity factor by

$$\frac{\Delta_t}{(\Gamma^e_t)^{1/\phi}} = \sum_z \psi_z \left(\frac{P_{zt}}{P_t}\right)^{-1}\left(\frac{\Delta_{zt}}{A_{zt}Q_{zt}}\right).$$

Γ^e_t denotes the aggregate growth trend that is derived below.

2 Planner Problem

To isolate the distortions in the allocation of the decentralized economy, we also derive the the first-best allocation from the planner problem. The solution of the planner problem involves the allocation of factor inputs across firms with different levels of productivity within the sector z; the allocation of factor inputs between sectors with different average productivities; and the optimal intertemporal paths of aggregate variables.

2.1 Sectoral and Aggregate Technologies

The within-sector allocation corresponds to the intratemporal allocation in the one sector model, when this allocation is applied to sector z. Thus, sectoral technology in the planned economy corresponds to

$$Y_{zt}^e = \frac{A_{zt} Q_{zt}}{\Delta_{zt}^e} \left((K_{zt}^e)^{1-\frac{1}{\phi}} (L_{zt}^e)^{\frac{1}{\phi}} - F_{zt} \right), \tag{10}$$

where the efficient level of the endogenous component of sectoral productivity is

$$1/\Delta_{zt}^e = \left(\int_0^1 \left(G_{jzt} Q_{t-s_{jzt}} / Q_{zt} \right)^{\theta-1} \mathrm{dj} \right)^{\frac{1}{\theta-1}},$$

and evolves according to equation (7).

To obtain the aggregate technology in the planner solution, the planner solves

$$\max_{Y_{zt}^e, L_{zt}^e, K_{zt}^e, \forall z} Y_t^e = \prod_z (Y_{zt}^e)^{\psi_z} \quad s.t. \quad Y_{zt}^e = \frac{A_{zt} Q_{zt}}{\Delta_{zt}^e} \left((K_{zt}^e)^{1-\frac{1}{\phi}} (L_{zt}^e)^{\frac{1}{\phi}} - F_{zt} \right),$$

with $L_t^e = \sum_z L_{zt}^e$ and $K_t^e = \sum_z K_{zt}^e$ and L_t^e, K_t^e given. The solution to this problem yields the aggregate technology

$$Y_t^e = \frac{(\Gamma_t^e)^{1/\phi}}{\Delta_t^e} \left((K_t^e)^{1-\frac{1}{\phi}} (L_t^e)^{\frac{1}{\phi}} - F_t \right), \tag{11}$$

with $F_t = \sum_z F_{zt}$ and defining

$$\frac{(\Gamma_t^e)^{1/\phi}}{\Delta_t^e} = \prod_z \psi_z^{\psi_z} \left(\frac{A_{zt} Q_{zt}}{\Delta_{zt}^e} \right)^{\psi_z}. \tag{12}$$

2.2 Intertemporal First-Best Allocation

The derivation of the intertemporal allocation in the planner problem proceeds along analogous steps as the derivation of the one-sector model. Therefore, the first-best allocation of aggregate variables implied by the planner solution corresponds to

$$
\begin{aligned}
(\Delta^e_{zt})^{1-\theta} &= \delta_z + (1-\delta_z)\left(\Delta^e_{zt-1} q_{zt}/g_{zt}\right)^{1-\theta} \\
\frac{(\Gamma^e_t)^{1/\phi}}{\Delta^e_t} &= \prod_z \psi_z^{\psi_z}\left(\frac{A_{zt}Q_{zt}}{\Delta^e_{zt}}\right)^{\psi_z} \\
Y^e_t &= \frac{(\Gamma^e_t)^{1/\phi}}{\Delta^e_t}\left((K^e_t)^{1-\frac{1}{\phi}}(L^e_t)^{\frac{1}{\phi}} - F_t\right), \\
Y^e_{Lt} &= -\frac{U^e_{Lt}}{U^e_{Ct}}, \\
1 &= \beta E_t\left[\frac{\xi_{t+1}}{\xi_t}\frac{U^e_{Ct+1}}{U^e_{Ct}}\left(Y^e_{Kt+1}+1-d\right)\right], \\
K^e_{t+1} &= (1-d)K^e_t + Y^e_t - C^e_t,
\end{aligned}
\tag{13}
$$

denoting by Y^e_{Kt} the marginal product of capital and by Y^e_{Lt} the marginal product of labor.

2.3 Balanced Growth Path

Let aggregate output grow with the trend Γ^e_t and sectoral output grow with the trend Γ^e_{zt}. The intertemporal feasibility condition implies that K^e_t grows at the same rate as aggregate output, and $K^e_t = \sum_z K^e_{zt}$ implies that K^e_{zt} grows at the same rate as K^e_t, i.e., at rate Γ^e_t.

We express sectoral and aggregate growth trends in terms of productivity parameters by using the sectoral and the aggregate technology in the planner solution. This yields

$$
\begin{aligned}
\Gamma^e_t &= \prod_{z=1}^{Z}\left(\frac{A_{zt}Q_{zt}}{\Delta^e_{zt}}\right)^{\psi_z\phi}, \\
\frac{\Gamma^e_{zt}}{\Gamma^e_t} &= \left(\frac{A_{zt}Q_{zt}}{\Delta^e_{zt}}\right) \Big/ \prod_{z=1}^{Z}\left(\frac{A_{zt}Q_{zt}}{\Delta^e_{zt}}\right)^{\psi_z}.
\end{aligned}
$$

These two equation further imply that

$$
\begin{aligned}
\gamma^e_t &= \prod_{z=1}^{Z}(\gamma^e_{zt})^{\psi_z}, \\
\gamma^e_{zt} &= (\gamma^e_t)^{1-\frac{1}{\phi}}\left(\frac{a_{zt}q_{zt}\Delta^e_{zt-1}}{\Delta^e_{zt}}\right),
\end{aligned}
$$

using $\gamma^e_{zt} = \Gamma^e_{zt}/\Gamma^e_{zt-1}$ and $\gamma^e_t = \Gamma^e_t/\Gamma^e_{t-1}$. We use the growth trends Γ^e_t and Γ^e_{zt} and the assumption that $F_{zt} = f_z(\Gamma^e_t)^{1-1/\phi}$ to convert the non-stationary variables in the system of equations (13) into stationary variables. This implies that the aggregate productivity factor in the planner solution is constant and corresponds to $1/\Delta^e = \prod_z \psi_z^{\psi_z}$.

3 Steady State in the Decentralized Economy

We use the sectoral and aggregate growth trend Γ_t^e and Γ_{zt}^e from the planner solution to also detrend the non-stationary variables in the decentralized economy. For this transformation, we define the stationary variables $y_t = Y_t/\Gamma_t^e$, $k_t = K_t/\Gamma_t^e$, $c_t = C_t/\Gamma_t^e$, and $w_t = W_t/(P_t\Gamma_t^e)$.

3.1 Steady State with Two Distortions

Then, we rewrite the detrended decentralized economy in a way that shows that only two distortions, the relative price distortion $\rho(\Pi)$ and the markup distortion $\mu(\Pi)$, which both depend on the aggregate inflation rate $\Pi_t = P_t/P_{t-1}$, prevent the decentralized economy from perfectly replicating the planner solution. These steps again are analogous to the steps in the one-sector economy (see Appendix A.8 in the paper). In the steady state, the decentralized multi-sector economy is then represented by the following equations.

$$y = \left(\frac{\rho(\Pi)}{\Delta^e}\right)\left(k^{1-\frac{1}{\phi}}L^{\frac{1}{\phi}} - f\right) \tag{14}$$

$$c\left(-\frac{V_L}{V(L)}\right) = \frac{\mu(\Pi)^{-1}}{\Delta^e}\left(\frac{1}{\phi}\right)\left(\frac{k}{L}\right)^{1-\frac{1}{\phi}} \tag{15}$$

$$1/[\beta\,(\gamma^e)^{-\sigma}] - 1 + d = \frac{\mu(\Pi)^{-1}}{\Delta^e}\left(1-\frac{1}{\phi}\right)\left(\frac{k}{L}\right)^{-\frac{1}{\phi}} \tag{16}$$

$$y = c + (\gamma^e - 1 + d)k. \tag{17}$$

Here, V_L denotes the derivative of $V(L)$. Given the aggregate distortions $\rho(\Pi)$ and $\mu(\Pi)$ and the aggregate growth rate γ^e, these equations determine y, k, L and c. The aggregate distortions are determined by the equations (suppressing the argument Π)

$$(\rho\mu)^{-1} = \sum_{z=1}^{Z}\psi_z(\mu_z\rho_z)^{-1}, \tag{18}$$

$$\mu = \prod_{z=1}^{Z}\mu_z^{\psi_z}, \tag{19}$$

and hence are defined in terms of sectoral distortions.

The sectoral relative price distortion is defined as $\rho_z = \Delta_z^e/\Delta_z$. Using the equations (7) and (8), which determine Δ_z^e and Δ_z, respectively, we can express the (inverse) sectoral relative price distortion in the steady state as a function of the aggregate inflation rate,

$$\rho_z(\Pi)^{-1} = \left(\frac{1-\alpha_z(1-\delta_z)(g_z/q_z)^{\theta-1}}{1-\alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta}(g_z/q_z)^{-1}}\right)\left(\frac{1-\alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta-1}}{1-\alpha_z(1-\delta_z)(g_z/q_z)^{\theta-1}}\right)^{\frac{\theta}{\theta-1}}, \tag{20}$$

which holds for $z = 1, \ldots Z$ and where we have used the fact that the sectoral inflation rate is related to the aggregate inflation rate according to $\Pi_z = (\gamma^e/\gamma_z^e)\,\Pi$, which follows from product demand $(P_{zt}/P_t) = \psi_z\,(Y_{zt}/Y_t)^{-1}$.

The sectoral markup distortion is defined as $\mu_z = p_z/mc$. Using the equations (3) to (5) and combining them with the definitions that $p_{zt} = (P_{zt}/P_t)\,(\Gamma^e_{zt}/\Gamma^e_t)$ and $mc_t = MC_t/(P_t(\Gamma^e_t)^{1/\phi})$, we can also express the sectoral markup distortion in the steady state as a function of the aggregate inflation rate,

$$\mu_z(\Pi) = \left(\frac{1}{1+\tau}\frac{\theta}{\theta-1}\right) \tag{21}$$

$$\left(\frac{1-\alpha_z(1-\delta_z)\beta(\gamma^e)^{1-\sigma}[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta-1}}{1-\alpha_z(1-\delta_z)\beta(\gamma^e)^{1-\sigma}[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta}(g_z/q_z)^{-1}}\right)\left(\frac{1-\alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta-1}}{1-\alpha_z(1-\delta_z)(g_z/q_z)^{\theta-1}}\right)^{\frac{1}{\theta-1}},$$

which holds for $z = 1, \ldots Z$ and where we have used that $\Pi_z = (\gamma^e/\gamma_z^e)\,\Pi$. To summarize, equations (14) to (21) represent the steady state equations that determine the variables $c, L, y, k, \mu, \rho, \mu_z, \rho_z$ for the sectors $z = 1, \ldots Z$ and given Π.

3.2 Conditions for the Existence of the Steady State

We provide existence conditions for the limiting case in which $\beta(\gamma^e)^{1-\sigma} \to 1$, which is the case for which we derive our main results in the multi-sector economy. First, we impose

$$1 > (1-\delta_z)\,(g_z/q_z)^{\theta-1}\,, \tag{22}$$

for all $z = 1, \ldots Z$, to ensure that Δ_z^e in equation (7) has a well-defined steady state value. Second, we impose conditions that ensure sectoral distortions in equations (20) and (21) that are well defined in the steady state with $\beta(\gamma^e)^{1-\sigma} \to 1$. These conditions are

$$1 > \alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta}(g_z/q_z)^{-1},$$
$$1 > \alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta-1}.$$

The additional condition $1 > \alpha_z(1-\delta_z)(g_z/q_z)^{\theta-1}$ is always fulfilled as a result of condition (22) and $\alpha_z < 1$.

4 Proof of Proposition 5

First, we show that under the conditions stated in the proposition, the sectoral relative price distortion and the sectoral markup distortion are inversely equal to each other and that therefore the two aggregate distortions are also inversely equal to each other. Second, we show that as a result, the aggregate steady state inflation rate that maximizes steady state utility can be derived by minimizing the aggregate markup distortion. Third, we show that this minimization yields the optimal aggregate steady state inflation rate in the proposition.

4.1 Proportionality of Distortions

In the steady state with $\beta(\gamma^e)^{1-\sigma} \to 1$ and $\frac{1}{1+\tau}\frac{\theta}{\theta-1} = 1$, the sectoral markup distortion in equation (21) can be rearranged according to

$$\begin{aligned}
\mu_z(\Pi) &= \left(\frac{1-\alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta-1}}{1-\alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta}(g_z/q_z)^{-1}}\right)\left(\frac{1-\alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta-1}}{1-\alpha_z(1-\delta_z)(g_z/q_z)^{\theta-1}}\right)^{\frac{1}{\theta-1}}, \\
&= \left(\frac{1-\alpha_z(1-\delta_z)(g_z/q_z)^{\theta-1}}{1-\alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta}(g_z/q_z)^{-1}}\right)\left(\frac{1-\alpha_z(1-\delta_z)[(\gamma^e/\gamma_z^e)\,\Pi]^{\theta-1}}{1-\alpha_z(1-\delta_z)(g_z/q_z)^{\theta-1}}\right)^{\frac{\theta}{\theta-1}}, \\
&= \rho_z(\Pi)^{-1},
\end{aligned}$$

where the last step follows from equation (20). Equation (18) then implies that

$$\mu(\Pi) = \rho(\Pi)^{-1}.$$

4.2 Steady State with One Distortion

The inverse relationship between the two aggregate distortions and $\beta(\gamma^e)^{1-\sigma} \to 1$ imply that equations (14) to (17) can be rearranged as

$$y = \left[\mu(\Pi)^{-1}/\Delta^e\right]\left(k^{1-\frac{1}{\phi}}L^{\frac{1}{\phi}} - f\right) \tag{23}$$

$$c\left(-\frac{V_L}{V(L)}\right) = \left[\mu(\Pi)^{-1}/\Delta^e\right]\left(\frac{1}{\phi}\right)\left(\frac{k}{L}\right)^{1-\frac{1}{\phi}} \tag{24}$$

$$\gamma^e - 1 + d = \left[\mu(\Pi)^{-1}/\Delta^e\right]\left(1-\frac{1}{\phi}\right)\left(\frac{k}{L}\right)^{-\frac{1}{\phi}} \tag{25}$$

$$y = c + (\gamma^e - 1 + d)k. \tag{26}$$

Accordingly, the sticky price economy consists of the definition of the aggregate markup in equation (19), the relationship $\mu_z(\Pi) = \rho_z(\Pi)^{-1}$ and equation (20) determining $\rho_z(\Pi)^{-1}$, and the equations (23) to (26).

In the proposition, we derive the aggregate steady state inflation rate Π that maximizes steady state utility subject to this sticky price economy. Given the structure of this economy, however, it turns out that instead of $\max_\Pi U(c(\Pi), L(\Pi))$, we can directly $\min_\Pi \mu(\Pi)$.

The reason for this is that Π enters equations (23) to (26) only through aggregate productivity $\mu(\Pi)^{-1}/\Delta^e$, which enters the aggregate technology in equation (23), the MPL in equation (24), and the MPC in equation (25). This implies that minimizing $\mu(\Pi)$

shifts the production possibility frontier of the social planner that seeks the optimal $\Pi^\star$ outwards and therefore also maximizes steady state utility.

4.3 Minimizing the Markup Distortion

Equation (19) implies that minimizing the aggregate markup distortion requires that

$$\frac{\partial \mu(\Pi)}{\partial \Pi} = \sum_{z=1}^{Z} \psi_z \mu_z(\Pi)^{\psi_z - 1} [\partial \mu_z(\Pi)/\partial \Pi] \left(\prod_{\neg z} \mu_z(\Pi)^{\psi_z} \right) = 0,$$

using $\neg z$ to denote the set of all sectors except sector z. Simplifying yields

$$\sum_{z=1}^{Z} \psi_z \frac{\partial \mu_z(\Pi)/\partial \Pi}{\mu_z(\Pi)} = 0. \tag{27}$$

We use equation (20), $\mu_z(\Pi) = \rho_z(\Pi)^{-1}$ and shorthand $s_z = \alpha_z(1-\delta_z)(\gamma^e/\gamma^e_z)^{\theta-1}$ to obtain

$$\frac{\partial \mu_z(\Pi)/\partial \Pi}{\mu_z(\Pi)} = \frac{\theta s_z \Pi^{\theta-2} \left(\frac{q_z \gamma^e}{g_z \gamma^e_z}\right)}{\left(1 - s_z \Pi^\theta \left(\frac{q_z \gamma^e}{g_z \gamma^e_z}\right)\right)\left(1 - s_z \Pi^{\theta-1}\right)} \left[\Pi - \left(\frac{q_z \gamma^e}{g_z \gamma^e_z}\right)^{-1}\right].$$

Plugging this expression into equation (27) and multiplying by Π^2 yields

$$\sum_{z=1}^{Z} \left(\frac{\psi_z \theta s_z \Pi^{\theta} \left(\frac{q_z \gamma^e}{g_z \gamma^e_z}\right)}{\left(1 - s_z \Pi^\theta \left(\frac{q_z \gamma^e}{g_z \gamma^e_z}\right)\right)\left(1 - s_z \Pi^{\theta-1}\right)} \right) \left[\Pi - \left(\frac{q_z \gamma^e}{g_z \gamma^e_z}\right)^{-1}\right] = 0. \tag{28}$$

We denote the weight in parenthesis by $\tilde{\omega}_z$, and normalize it so that it sums to unity. This yields the new weight $\omega_z = \tilde{\omega}_z / \sum_{z=1}^{Z} \tilde{\omega}_z$, with $\sum_{z=1}^{Z} \omega_z = 1$. Thus, we obtain

$$\sum_{z=1}^{Z} \omega_z \left[\Pi^\star - \left(\frac{g_z \gamma^e_z}{q_z \gamma^e}\right)\right] = 0, \tag{29}$$

where ω_z is given by the expression in the proposition. Solving equation (29) for $\Pi^\star$ also yields the optimal aggregate steady state inflation rate in the proposition.

5 Proof of Lemma 3

To derive the lemma, we denote $m_z = \frac{g_z \gamma^e_z}{q_z \gamma^e}$ and repeat equation (28) with the new notation:

$$\sum_{z=1}^{Z} \tilde{\omega}_z(\Pi, m_z) \left[\Pi - m_z\right] = 0, \tag{30}$$

with $\tilde{\omega}_z(\Pi, m_z) = \frac{\psi_z \theta s_z \Pi^\theta / m_z}{\left(1 - s_z \Pi^\theta / m_z\right)\left(1 - s_z \Pi^{\theta-1}\right)}$ and $s_z = \alpha_z(1-\delta_z)(\gamma^e/\gamma^e_z)^{\theta-1}$. Expanding equation (30) accurate to the first order at the points $\bar{\Pi}$ and $\bar{m}_z$, with $\bar{\Pi} = \bar{m}_z$, yields

$$\sum_{z=1}^{Z} \tilde{\omega}_z(\bar{\Pi}, \bar{m}_z) \left[\Pi - m_z\right] = 0 + O(2).$$

Rewriting this equation yields

$$\Pi^\star = \left(\sum_{z=1}^{Z} \tilde{\omega}_z(\bar{\Pi}, \bar{m}_z) \right)^{-1} \sum_{z=1}^{Z} \tilde{\omega}_z(\bar{\Pi}, \bar{m}_z)\ m_z + O(2). \tag{31}$$

$\Pi^\star$ is a weighted average of the m_z's for all sectors z and with weights evaluated at the expansion point and normalized to unity. The normalized weight of sector z evaluated at $\bar{\Pi} = \bar{m}_z$ corresponds to

$$\begin{aligned} \frac{\tilde{\omega}_z(\bar{\Pi}, \bar{m}_z)}{\sum_{z=1}^{Z} \tilde{\omega}_z(\bar{\Pi}, \bar{m}_z)} &= \psi_z \left[\frac{\theta s_z \bar{\Pi}^{\theta-1}}{\left(1 - s_z \bar{\Pi}^{\theta-1}\right)^2} \right] \left(\sum_{z=1}^{Z} \psi_z \left[\frac{\theta s_z \bar{\Pi}^{\theta-1}}{\left(1 - s_z \bar{\Pi}^{\theta-1}\right)^2} \right] \right)^{-1}, \\ &= \psi_z, \end{aligned}$$

where the second equality follows from the requirement in the lemma that $s_z = \alpha_z(1 - \delta_z)(\gamma^e/\gamma_z^e)^{\theta-1}$ is the same for all sectors $z = 1, \ldots Z$ and the fact that $\sum_{z=1}^{Z} \psi_z = 1$. Hence,

$$\Pi^\star = \sum_{z=1}^{Z} \psi_z m_z + O(2),$$

which corresponds to the equation in the lemma after using $m_z = \frac{g_z \gamma_z^e}{q_z \gamma^e}$.

6 Proof of Proposition 6

To derive the proposition, we solve technology $Y_{jzt} = A_{zt} Q_{z,t-s_{jzt}} G_{jzt} \left(K_{jzt}^{1-1/\phi} L_{jzt}^{1/\phi} - F_{zt} \right)$ of the firm j in sector z for labor L_{jzt}. The fact that the optimal capital labor ratio of firm j corresponds to the aggregate capital labor ratio yields

$$L_{jzt} = \left(\frac{K_t}{L_t} \right)^{\frac{1}{\phi}-1} \left(F_{zt} + \frac{Y_{jzt}}{A_{zt} Q_{z,t-s_{jzt}} G_{jzt}} \right). \tag{32}$$

We now replace Y_{jzt} by the firm's product demand $Y_{jzt} = \left(P_{jzt}/P_{zt}\right)^{-\theta} Y_{zt}$. To express the relative price in product demand by the relative productivity, we use the pricing equations (3) to (5), impose $\alpha_z = 0$, and combine them with the pricing equations for a firm in sector z that receives a δ shock in period t. This yields

$$\frac{P_{jzt}}{P_{zt}} \left(\frac{Q_{zt-s_{jzt}} G_{jzt}}{Q_{zt}} \right) = \frac{P_{z,t,t}^\star}{P_{zt}}. \tag{33}$$

Using the notation $p_{zt}^\star = P_{z,t,t}^\star / P_{zt}$, we obtain from the equation (6) with $\alpha_z = 0$ that $p_{zt}^\star = 1/\Delta_{zt}^e$. Accordingly, we rearrange equation (33) to obtain

$$\frac{P_{jzt}}{P_{zt}} = \left(\frac{A_{zt} Q_{zt}}{\Delta_{zt}^e} \right) \frac{1}{A_{zt} Q_{z,t-s_{jzt}} G_{jzt}}.$$

Plugging this equation for the relative price into product demand yields

$$Y_{jzt} = (\Delta^e_{zt})^\theta \left(\frac{Q_{z,t-s_{jzt}} G_{jzt}}{Q_{zt}} \right)^\theta Y_{zt},$$

or

$$\begin{aligned} \frac{Y_{jzt}}{A_{zt} Q_{z,t-s_{jzt}} G_{jzt}} &= \left[\left(\frac{A_{zt} Q_{zt}}{\Delta^e_{zt}} \right) \left(\frac{Q_{z,t-s_{jzt}} G_{jzt}}{Q_{zt}} \right) \right]^{-1} (\Delta^e_{zt})^{\theta-1} \left(\frac{Q_{z,t-s_{jzt}} G_{jzt}}{Q_{zt}} \right)^\theta Y_{zt}, \\ &= \left[\Gamma^e_{zt} (\Gamma^e_t)^{\frac{1}{\phi}-1} \right]^{-1} (\Delta^e_{zt})^{\theta-1} \left(\frac{Q_{z,t-s_{jzt}} G_{jzt}}{Q_{zt}} \right)^{\theta-1} Y_{zt}, \end{aligned}$$

or

$$\frac{Y_{jzt}}{A_{zt} Q_{z,t-s_{jzt}} G_{jzt}} = \left(Q_{z,t-s_{jzt}} G_{jzt} / Q_{zt} \right)^{\theta-1} y_{zt} (\Delta^e_{zt})^{\theta-1} (\Gamma^e_t)^{1-\frac{1}{\phi}}, \tag{34}$$

with $y_{zt} = Y_{zt}/\Gamma^e_{zt}$ and since trend growth in the multi-sector economy implies that $\Gamma^e_{zt}(\Gamma^e_t)^{\frac{1}{\phi}-1} = A_{zt} Q_{zt}/\Delta^e_{zt}$. Substituting equation (34) into equation (32) and using $k_t = K_t/\Gamma^e_t$ and $F_{zt} = f_z (\Gamma^e_t)^{1-\frac{1}{\phi}}$ yields

$$L_{jzt} = \left(f_z + \left(Q_{z,t-s_{jzt}} G_{jzt} / Q_{zt} \right)^{\theta-1} y_{zt} (\Delta^e_{zt})^{\theta-1} \right) \left(\frac{k_t}{L_t} \right)^{\frac{1}{\phi}-1}, \tag{35}$$

which shows that L_{jzt} grows with the relative productivity $Q_{z,t-s_{jzt}} G_{jzt}/Q_{zt}$. Imposing zero fixed costs $f_z = 0$ and taking the natural logarithm yields

$$\ln(L_{jzt}) = (\theta - 1) \ln \left(Q_{z,t-s_{jzt}} G_{jzt}/Q_{zt} \right) + d_{zt}.$$

The composite variable $d_{zt} = \ln \left((k_t/L_t)^{\frac{1}{\phi}-1} y_{zt} (\Delta^e_{zt})^{\theta-1} \right)$ and varies with time t and sector z. Dropping the sector subscript z to economize on notation thus yields

$$\ln(L_{jt}) = (\theta - 1) \ln \left(Q_{t-s_{jt}} G_{jt}/Q_t \right) + d_t. \tag{36}$$

Given our assumptions on the productivity processes, we have

$$\begin{aligned} \ln \left(Q_{t-s_{jt}} G_{jt}/Q_t \right) &= \ln \left(\frac{Q_0 q^{t-s_{jt}} g^{s_{jt}}}{Q_0 q^t} \right) + \ln \left(\prod_{i=1}^{t-s_{jt}} \varepsilon^q_i \right) \left(\prod_{i=t-s_{jt}+1}^{t} \varepsilon^g_i \right) / \left(\prod_{i=1}^{t} \varepsilon^q_i \right), \\ &= s_{jt} \ln(g/q) + \frac{1}{\theta - 1} \epsilon_{jt}, \end{aligned} \tag{37}$$

where $\epsilon_{jt} = (\theta - 1) \ln \left(\prod_{i=t-s_{jt}+1}^{t} \varepsilon^g_i \right) / \left(\prod_{i=t-s_{jt}+1}^{t} \varepsilon^q_i \right)$ is a stationary residual under the stated assumptions. Plugging equation (37) into equation (36) delivers the equation in the proposition.